Figuring Out: The Spread of Hindu-Arabic Numerals in the European Tradition of Practical Mathematics (13^{th}-16^{th} centuries)

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Abstract: The paper contributes to the literature focusing on the role of ideas, practices and human capital in pre-modern European economic development. It argues that studying the spread of Hindu-Arabic numerals among European practitioners allows to open up a perspective on a progressive transmission of useful knowledge from the commercial revolution to the early modern period. The analysis is based on an original database recording detailed information on over 1200 texts, both manuscript and printed. This database provides the most detailed reconstruction available of the European tradition of practical arithmetic from the late 13th to the end of the 16th century. It can be argued that this is the tradition which drove the adoption of Hindu-Arabic numerals in Europe. The dataset is analysed with statistical and spatial tools. Since the spread of these texts is grounded on inland patterns, the evidence suggests that a continuous transmission of useful knowledge may have played a role during the shift of the core of European trade from the Mediterranean to the Atlantic.

Introduction

Given their almost universal diffusion, Hindu-Arabic numerals can easily be taken for granted. Nevertheless, for a long stretch of European history numbers have been represented using Roman numerals, as the positional numeral system reached western Europe at a relatively late stage. This paper provides new evidence to reconstruct the transition of European practitioners from Roman to Hindu-Arabic numerals. Moreover, it argues that the study of this phenomenon provides a proxy to chart the diffusion of the financial and organisational innovations first developed with the commercial revolution,[1] as Hindu-Arabic arithmetic provided the mathematical underpinning of such innovations.

The origins and spread of Hindu-Arabic numerals from 5th century India have been the object of a considerable amount of research (Smith & Karpinski 1911), (Hill 1915), (Ifrah 2000), (Chrisomalis 2010). Scholars have also focused on the complex introduction of Hindu-Arabic numerals in Europe from the 10th to the 13th century, as summarised in (Burnett 2010). Yet, the spread of Hindu-Arabic numerals within Europe has received limited scholarly attention, the only recent work on the subject being (Schärlig 2010). This paper contributes to this field by providing a detailed reconstruction of the continuous and European-wide tradition of practical arithmetic from the late 13th century to 1600, based on an original database recording 1280 texts written by over 340 authors. Thanks to this tradition, the use of Hindu-Arabic numerals spread among merchants and practitioners, who needed such mathematics for their commercial practice. Since it provided the best tools to deal with multiplications and divisions (fractions and the positional numeral system), Hindu-Arabic mathematics was adopted by the first sedentary merchants, who, being increasingly involved in

[1] I understand 'commercial revolution' in its stricter sense, as originally formulated by (De Roover 1953), that is to say a "a complete or drastic change in the methods of doing business or in the organization of business enterprise" which occurred in the 13th century. This drastic change led to the emergence of the first sedentary merchants and to the introduction of breakthrough innovations for European business history, such as the bill of exchange, permanent partnerships, and double entry bookkeeping.

international trade, faced a growing need to calculate multiplications and divisions in the form of conversions and exchange rates. The bill of exchange, which revolutionised the European international monetary market, was developed in the same time and the same area where the first tradition of practical arithmetic flourished. As the bill of exchange was grounded on ratios of exchange rates, Hindu-Arabic mathematics arguably played an important role in the development of this financial innovation.

The paper also contributes to the literature focusing on the role of ideas, practices and human capital in pre-modern European economic development. Qualitative and quantitative studies have documented the role of upper tail human capital in the industrial revolution (Mokyr 2005), (Mokyr 2011), (Squicciarini & Voigtländer 2015). Drawing from classical works (Sombart 1928), (Sombart 1953), a recent literature is investigating the role of human capital during the pre-modern period as well. Providing estimates of European book production (Baten & van Zanden 2008) and (Van Zanden 2009) argue that it is possible to observe an emerging "knowledge economy" starting from the late middle ages. This paper elaborates on this thesis providing evidence grounded on clearly identifiable sources and focussing on a particularly significant cross section of overall book production.

(Mokyr 2017) stresses the importance for economic development of the growing diversity and permeability of the European "market for ideas" from the beginning of the early modern period. (Dittmar 2011) has documented the significant role of such ideas identifying a causal relationship between the spread of the printing press and urban growth. More recently, Dittmar and Seabold have narrowed their focus by isolating the impact of the publication of business manuals – which consist mostly of practical arithmetics – on economic development (Dittmar & Seabold 2019). Their econometric analyses find no robust relationship between other kinds of publications and city growth, a causal effect of the spread of business manuals on urban growth, and a strong correlation with individual achievement in bourgeois occupations between 1500 and 1600.

These findings provide a strong background for the argument of the present paper. However, they take into consideration only printed sources. While this is due to the quantity, quality and increasing accessibility of early printed sources, these analyses have the limit of not considering intellectual developments predating the introduction of the printing press. In other words, such an approach is exposed to the risk of reducing histories of knowledge and skill transfers to the history of their diffusion within one technology – i.e. the printing press. While the impact of printing clearly played a major role in lowering access costs and in accelerating the diffusion of ideas, we should not forget that those very ideas had already been circulating and influencing social and economic phenomena for a long time. This paper contributes to this debate providing evidence based on both printed and manuscript material.

The main consequence of analysing both manuscript and printed sources is that the picture of the diffusion of the innovations of the commercial revolution – understood as specified above – changes considerably. Statistical and geographical analyses of the data show a transmission of practical arithmetic that followed a marked south-to-north axis. This pattern, which is remarkably different from that of the diffusion of the printing press, documents a progressive transmission of knowledge from the first centres of the commercial revolution to the rest of Europe. This process can be conceived of as a slow innovation cycle which took more than 2.5 centuries to complete.

England, where the publication of practical arithmetics took off in the second half of the 16th century, and where Hindu-Arabic numerals still had a limited circulation at the end of the century, stands out as the latecomer in this cycle. Interestingly, the spread of practical arithmetic followed an inverse path from that of successive processes of political and economic 'emulation' characteristic of the early modern period (Reinert 2011). To capture the market- and practice-oriented nature of the diffusion of these texts, I employ the concept of 'useful knowledge' as given by (Mokyr 2005) to describe the tradition of practical arithmetic texts, suggesting the possibility of extending Mokyr's framework to the onset of pre-modern European economic development.

Finally, the spread of practical arithmetic was clearly grounded on inland networks. This is an interesting finding which may raise the question as to whether a relevant contribution to the transition of the core of European trade from the Mediterranean to the Atlantic – the premise of the so-called "first great divergence" (Acemoglu *et al.* 2003) – came not only from maritime trade, but also from inland and intellectual exchanges, i.e. from the spread of useful knowledge and skills.

The paper is organised in four sections. The first section is dedicated to explaining the adoption of Hindu-Arabic numerals during the commercial revolution. The second section describes the European tradition of practical arithmetic, providing both qualitative evidence and a statistical and geographic analysis of the dataset. Section three shows that the spread of practical arithmetic correlated with the adoption of Hindu-Arabic numerals in commercial documentation. Section four wraps up and concludes.

I.

Like other techniques imported from the East, such as the manufacturing of paper, glass and silk (Abulafia 1997), Hindu-Arabic numerals were first developed in the East – probably in 5th century India – and were brought to Europe across the Mediterranean during the late medieval period, following a long-run pattern of East-West exchanges across the Eurasian landmass (Diamond 1997), (Ifrah 2000), (Kunitzsch 2003), (Rashed 2011). Like other technologies imported from the Levant, Hindu-Arabic mathematics was re-contextualised and adapted to European needs. In 1202 Pisa, Leonardo Pisano (better known as Fibonacci) completed his *Liber abaci* (Franci 2002). This merchant-mathematician presented his work as the result of the studies he had carried out while travelling for business reasons to Maghreb and other Mediterranean ports. It was a ground-breaking work not so much for its original contributions, but rather for its role in summarising and spreading Hindu-Arabic mathematics. Written in Latin, the *Liber Abaci* showed the functioning of the positional numeral system, presented a thorough summary of Hindu-Arabic mathematics of the time and included several sections on how such mathematics could have been used to solve practical and commercial problems (Giusti & Petti 2002). The *Liber Abaci* is a sophisticated mathematical work which cannot be reduced to a commercial arithmetic, but its practical component had a fundamental influence on successive developments in the applications of mathematics.

The 13th century was the heyday of Mediterranean trade and of maritime republics (Lane 1973), (Findlay & O'Rourke 2009). The financial and organizational innovations of the commercial revolution, such as permanent partnerships (the *compagnia* and the *accomandita*), the European bill of exchange and double-entry bookkeeping were developed in Italian city-states by the first European sedentary merchants (Melis 1950), (De Roover 1953), (De Roover 1954), (Chaudhury & Denzel 2008), (Goldthwaite 2009). Together with the development of these innovations, merchant-bankers were the first European practitioners to use – alongside the previous Roman notation – Hindu-Arabic numerals.

The positional numeral system presents several advantages over the Roman one. Since it allows to handle very high numbers, it is more powerful than the Roman one. It does not require a separate reckoning tool, such as a reckoning table or other counting devices. It allows to check a calculation without repeating the entire procedure. Late medieval Arabic mathematics also provided the first step in what would have become modern mathematical symbolism. It introduced the symbolism we still employ to represent fractions, which allowed to handle proportions, multiplications and divisions more effectively. However, the written character of Arabic mathematics calls for a community of literate users, a rather high requirement for late medieval and early modern practitioners (Otis 2017).

Since they had to rely on written communication in order to coordinate with their agents abroad, sedentary merchants became literate during the 13th century.[2] The growing scale of their international business brought them to progressively handle higher numbers and to increasingly deal with conversions between measures and currencies. It seems that the bill of exchange in its fully-fledged form was developed by the end of the 13th century, and that by the first half of the 14th century it had become the main instrument to carry out international payments (Spufford 1988), (Chaudhury

[2] As early as 1201, Boncompagno da Signa, professor of *ars dictaminis* in Bologna, noticed that these merchants continuously "wrote and rewrote" letters among themselves in either a "corrupted Latin" or directly in their own vernaculars (Folena 1990, p. 229).

& Denzel 2008). The net quantity of precious metal transported between the main European trading centres did not decrease with the introduction of the bill of exchange, but "the amount of business that it represented was increased out of all proportion", as it "enormously multiplied the supply of money available for international transactions" (Spufford 1988, pp. 254-255).

Together with providing the main tool to carry out cashless international payments, the bill of exchange was also used as an instrument of credit, as it allowed to conceal a credit at interest under a double operation of exchange, bypassing the church's ban on interest-bearing loans (De Roover 1954). This is the key feature that separates the European bill of exchange from its Islamic homologue, as it seems that the Islamic bill of exchange was not used as an instrument of credit (Rubin 2010). Since the actual interest rate of the loan depended on the relative movement of exchange rates across two markets, lenders needed to anticipate exchange rates trends. The Venetian monetary market offered a suitable context to carry out such operations, as the seasonal departure of the state-owned galley fleets determined regular trends in the demand for liquidity and therefore foreseeable peaks and troughs in the quotation of money (Mueller 1997, chapter 8, Appendix C) (Bell *et al.* 2017), (Li 2017). While the first experimentations of the credit function of the bill of exchange could have already occurred in the middle of the 13th century, they certainly came into use by the early 14th, as in 1301 the Venetian government instructed the consuls of the merchants "not to register any except licit exchanges" (Spufford 1986, p. xliv). Florentine merchant-bankers exploited their strategic position and specialised in these operations, to the point that they held a "virtual monopoly of exchange" in the Rialto from the first half of the 14th century (Mueller 1997, p. 256).

These developments corresponded with the adoption of Hindu-Arabic numerals by Tuscan merchant-bankers (Van Egmond 1976). The growing scale of international trade and the central role given to exchange by the development of the bill of exchange led to an increasing need to calculate exchange and interest rates. This emerging need made the gains associated with adopting the positional numeral system outweigh its costs, as Hindu-Arabic numerals provided a more effective mathematical notation to deal with multiplications and divisions. The changing nature of their business made Tuscan merchant-bankers adopt in their practice the mathematical instruments which had been introduced by Fibonacci in the beginning of the century.[3] During this period of financial and organisation experimentation, the use of a technology imported from the Levant was re-contextualised as part of a new framework. In the case of the bill of exchange, Hindu-Arabic mathematics provided the mathematical underpinning of a fundamental change in international finance.

As Hindu-Arabic mathematics became a requirement, it was necessary to provide would-be international merchant-bankers with the necessary training in such science. Starting from the late 13th century, Italian commercial centres developed a tradition of practical arithmetic that matched such need. Following the terminology of primary sources, this tradition of Italian practical arithmetic is known as 'abacus mathematics'. This is a misleading label, as abacus treatises show almost exclusively Hindu-Arabic numerals and never illustrate how to make calculations with a reckoning table. Like the other mercantile sources of the period, abacus treatises are overwhelmingly written in the Italian vernaculars.

Abacus mathematics developed within the *ad hoc* institutional framework of 'abacus schools', i.e. both private and public schools, independent from universities, where 'abacus masters' taught children aged between 11 and 13 the fundamentals of Hindu-Arabic mathematics. Abacus masters were also the main authors of abacus manuals. We have direct archival evidence of a considerable diffusion of abacus schools in central-northern Italian cities.[4] Florence, the capital of international commercial-banking, was not surprisingly also the most important centre for abacus mathematics, with documented evidence of more than 70 abacus masters active in over 20 abacus schools (Ulivi 2008).

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[3] The argument here is not that Hindu-Arabic numerals replaced previous mathematic techniques, but rather that the adoption of Hindu-Arabic numerals opened possibilities that were hampered by previous systems. Italian late medieval merchants did not stop using Roman numerals as well as other reckoning tools. They had more tools to resort to and probably switched from one to the other technique depending on convenience.

[4] By the 15th century, abacus schools were founded in Verona, Venice, Florence, Bologna, Siena, Perugia, Palermo, Arezzo, Pisa, Volterra, Colle Val d'Elsa, Lucca, Milan, Pistoia, Prato, Fucecchio, Genoa, Savona, and Città di Castello (Ulivi 2008).

Abacus masters competed among each other to secure pupils and to solve mathematical problems of a growing complexity, acting also as consultants to merchant-bankers (Ulivi 2015). Already in 14th century Florence almost every practitioner had attended an abacus school, and in 15th century Italian commercial cities the diffusion of this mathematics was widespread among economic agents (Goldthwaite 1972), (Grendler 1989), (Goldthwaite 2009, chapter V). Abacus schools were attended not only by would-be merchant bankers, but also by engineers, architects, artisans and artists.[5]

Among a rich degree of variation, these texts spread a standard package of mathematical knowledge. Although a degree of progress in mathematical theory was achieved during this period – the most important being the first steps towards modern algebraic notation – the main contribution of this tradition was to spread the mathematics that had been introduced in the time of Fibonacci. Together with the introduction of the positional numeral system and of fractions, these texts are founded on the so-called "rule of three", i.e. the calculation of the unknown term of a proportion where three terms are known. What follows are the first lines of the earliest surviving abacus manual (c. 1290):

> Quisto è-ne lo livero de l'abbecho secondo la oppenione de
> maiestro Leonardo de la chasa degl figluogle Bonaçie da
> Pisa. Lo p(r)imo chapitolo è-ne de le reg(o)le d(e) le tre chose.
> Se ce fosse dicta alchuna ragione e · lla q(ua)-
> le se proponesse tre chose, sì devemo m(ultip)licare
> quilla chosa che noie volemo sap(er)e (contra) q(ui)lla
> che non è de quilla medessme, a pa(r)ti(r)e nell'a-
> ltra.[6]

[5] For example, we know that Filippo Brunelleschi, Leonardo da Vinci, Leon Battista Alberti, Piero della Francesca and Michelangelo attended an abacus school. Abacus mathematics was fundamental for the development of linear perspective. Piero della Francesca was so versed in abacus mathematics that he was also the author of an advanced abacus manual (Piero della Francesca 1970), (Gamba *et al.* 2012).

[6] This is the *Livero de l'abbecho* [the book of the abacus] according to the opinion of master Leonardo of the house of the sons of Bonaccio [*filii Bonaccii*] of Pisa. The first chapter is (about) the rule of three. / If we were told any question where three things are given, we have to multiply the thing we want to know by the thing that is not of the same kind, and

As shown in this passage, the main focus of these texts is not on mathematical theory, but on the pragmatic relevance of mathematics. Theoretical explanations, when present, are given a very limited space. The relevant mathematical rule is generally given without providing a demonstration, and is followed by a list of practical examples where the rule is applied.

Most of these texts are in fact composed of lists of solved practical problems, arranged according to their context of application, such as exchange, conversion, interest rates, divisions of shares, and alloying. This is a typical example, taken from the most successful abacus treatise of the 14th century (Florence, c. 1339):

> Quest'è una regola la quale mostra gieneralmente per tutte terre e da tutti pesi e da tutte monete. Quando noi comparassimo argento a biglone o d'oro, o sseta o spezierie o grascia o qualunque altro avere di peso fosse e noi lo portassimo da una terra ad un'altra, di sapere quanto mi chonviene vendere lo marcho o l'oncia o la libra o 'l quintale o la charica o qualunque altro peso fosse a salvarmi della vendita e riavere mio chapitale al peso ed alla moneta. Fa chosie: sempre multiplicha lo peso della terra onde tu lo trai chontra la valuta di quello luogho. E parti per la quantitade del peso oue tu lo porti sieno oncie o libre o chosse fosse.
>
> Verbi grazia lo marcho dell'argento di chorte lo qual è 8 oncie mi chosta 75 soldi della moneta. Portolo a gienova e questo marcho mi torna 8 oncie e quarta. Dimmi per quanto posso dare lo marcho in gienova e ssalvarmi. Dovemo multiplicare 8 via 75 soldi fanno 600 soldi et partire per 8 et $\frac{1}{4}$ sechondo la reghola che ne viene 72 soldi e 8 denari e $\frac{8}{11}$. Et per tanto possiamo dare lo marcho a gienova della moneta di vignone cioè 72 soldi 8 denari $\frac{8}{11}$.[7]

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divide into the other. Florence, Biblioteca Riccardiana, ms. 2404, f. 1r, 1-8, edited in (Bocchi 2017, p. 163). English translation by the author.

[7] This is a rule which can be applied [literally: is shown] to every city, every weigh and every currency. When we compare silver or bullion or gold, or silk or spices or provisions or any other kind of goods sold by weigh, and we carry such goods from one city to another, [we want to] know at what price it is convenient to sell by the mark or ounce or pound or quintal or *charica* or any other weight in order to be refunded [literally: be saved] from the selling and regain our capital. Do as follows: always multiply the weight of the city you are shipping from by the currency of that city. And divide into the weight of the city you are shipping to, whatever this may be. For example, the mark of silver of the court [the papal court then in Avignon], which is 8 ounces, costs 75 soldi of money of Avignon. I take it to Genoa, where the mark of silver weights 8 ounces and a quarter. Tell me at what price I can sell the mark in Genoa in order to be refunded. We have to multiply 8 by 75 soldi, which is 600 soldi, and divide 600 into 8 and $\frac{1}{4}$, according to the rule, which results in 72 soldi, 8 denari and $\frac{8}{11}$. And this is the price we can sell in Geona the mark of Avignon, i.e. 72 soldi, 8 denari and $\frac{8}{11}$. Paolo dell'Abaco, *Trattato di tutta l'arte dell'abacho*, Florence, Biblioteca Nazionale Centrale, ms. Fondo Nazionale II. IX. 57, fol. 53 r. English translation by the author.

As shown in this passage, this simple arithmetic was fundamental to solve problems of exchange. It is interesting to notice the dialogical structure of the text, where the writer directly addresses the reader ('do as follows', 'tell me'), which is derived from the teaching practice of abacus schools, where masters addressed their students.[8] Although they do not directly explain the functioning of the bill of exchange – whose details, in the same way as bookkeeping, were probably learnt during successive apprenticeships – abacus manuals provided the mathematical tools necessary to handle them. The following example, taken from an early 14th century Pisan manual, asks to compare exchange rates in order to know which currency is best to use to transfer money:

> Pagamento da una città a un'altra
>
> Uno ha fare uno pagamento a bologna di lb 500 di bolognini e truova che il bolognino grosso vale in bologna d 12 vale in firenze d 20. E llo grosso guelfo che vale in firenze d 48 vale in bologna d 30 e l'ancontano grosso che vale in firenze d 40 vale in bologna d 25. Adomando quale di queste 3 monete sarà milliore a portare e quanto.[9]

Italian abacus mathematics was a forerunner in Europe. The production of practical arithmetic texts employing Hindu-Arabic numerals outside of Italy consolidated only in the 15th century. Abacus schools predated the development of practical arithmetic schools teaching Hindu-Arabic numerals in the rest of Europe. Like in Italy, the European tradition of practical arithmetic was not dominated by the Latin language and was accessible to a relatively wide readership. As with abacus mathematics, also in the European tradition the focus was not on mathematical theory, but rather on its application to the solution of practical problems mainly through applications of the rule of three. European

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[8] Robert Recorde's *The ground of artes* (1543 London), the most important practical arithmetic manual of 16th century England, reproduces this dialogical structure.

[9] Payment from one city to the other. / One has to make a payment in Bologna of 500 pounds of bolognini, and they find that the bolognino grosso is worth 12 denari in Bologna and 20 denari in Florence. And the Guelf grosso, which is worth 48 denari in Florence, is worth 30 denari in Bologna. And the grosso of Ancona, which is worth 40 denari in Florence, is worth 25 denari in Bologna. I ask which of these three currencies will be best [to make the payment] and by how much. Florence, Biblioteca Riccardiana, ms. 2252, fol. 16 r.

practical arithmetic was a continuous tradition largely stemming from abacus mathematics. To my knowledge, such a tradition has never been studied in a comprehensive manner and has never been reconstructed as a continuous and European-wide phenomenon.

II.

The analysis presented in this section is based on an original database reconstructing the European tradition of practical arithmetic. (Van Egmond 1980), the most complete catalogue available on the tradition of abacus mathematics, identifies 288 manuscript and 153 printed texts composed from the 13th century to 1600: a tradition of remarkable size in late medieval and early modern terms. My dataset expands this approach to the main areas of western Europe relying both on secondary sources and on extensive archival research. I have recorded detailed information on 1280 practical arithmetic texts written from the *Liber Abaci* (1202) to 1600. Of these texts, 342 are manuscripts and 938 are printed. Following (Van Egmond 1980), every manuscript document has been considered as an independent text. As in (USTC) and (Hoock *et al.* 1991), I have considered printed texts at the book-edition level. The database records over 340 authors who wrote in more than 130 cities in most European languages.[10] All the recorded texts employ Hindu-Arabic numerals, and most of them use arithmetic as a commercial tool. The introduction of the printing press imposed a higher degree of standardisation on these manuals, which became the ancestors of modern mathematical primers. The dataset is described in detail in the Data Appendix.

The tradition of practical arithmetic is particularly suitable for a long-run analysis because its size is considerably bigger and its preservation is less scattered than other traditions of commercial writings – such as mercantile handbooks, accounting treatises and account books. (Spufford 2002) lists only 22 late medieval manuscript “merchant notebooks” while the Italian abacus tradition lists more than 280 manuscript texts. In relation with bookkeeping manuals, practical arithmetic

[10] Several texts are written by anonymous authors, hence the estimate on their number.

constitutes their premise for two reasons: because basic arithmetic is necessary to keep accounts and because, as a consequence, its spread occurred earlier than that of bookkeeping treatises. Moreover, neither the tradition of bookkeeping manuals is comparable to that of practical arithmetic in terms of scale, as, despite having been the focus of important scholarly efforts, the most comprehensive edition of early bookkeeping manuals includes only 31 texts (Yamey 1989). In comparison with the direct study of accounting sources (i.e. of account books), the tradition of European practical mathematics provides a continuous source of evidence, while the preservation of account books is very scattered. Furthermore, this tradition follows quite closely the path of the diffusion of modern accounting as investigated by accounting historians.

The size of the sample allows to identify some interesting trends. From a general point of view, it is possible to observe an exponential increase in the overall production of practical arithmetic manuals across the observation period, with a marked acceleration occurring in the 16th century, as a consequence of the introduction of the printing press and of the diffusion of the tradition across Europe. In **Figure 1**, which allows to observe both the overall data and the regional contribution for each interval, it is possible to notice the leading role played by the Italian abacus tradition until the first half of the 15th century. Starting from this date, there was a marked expansion in the production of texts across all other European areas, while the Italian production levelled off.

Figure 1

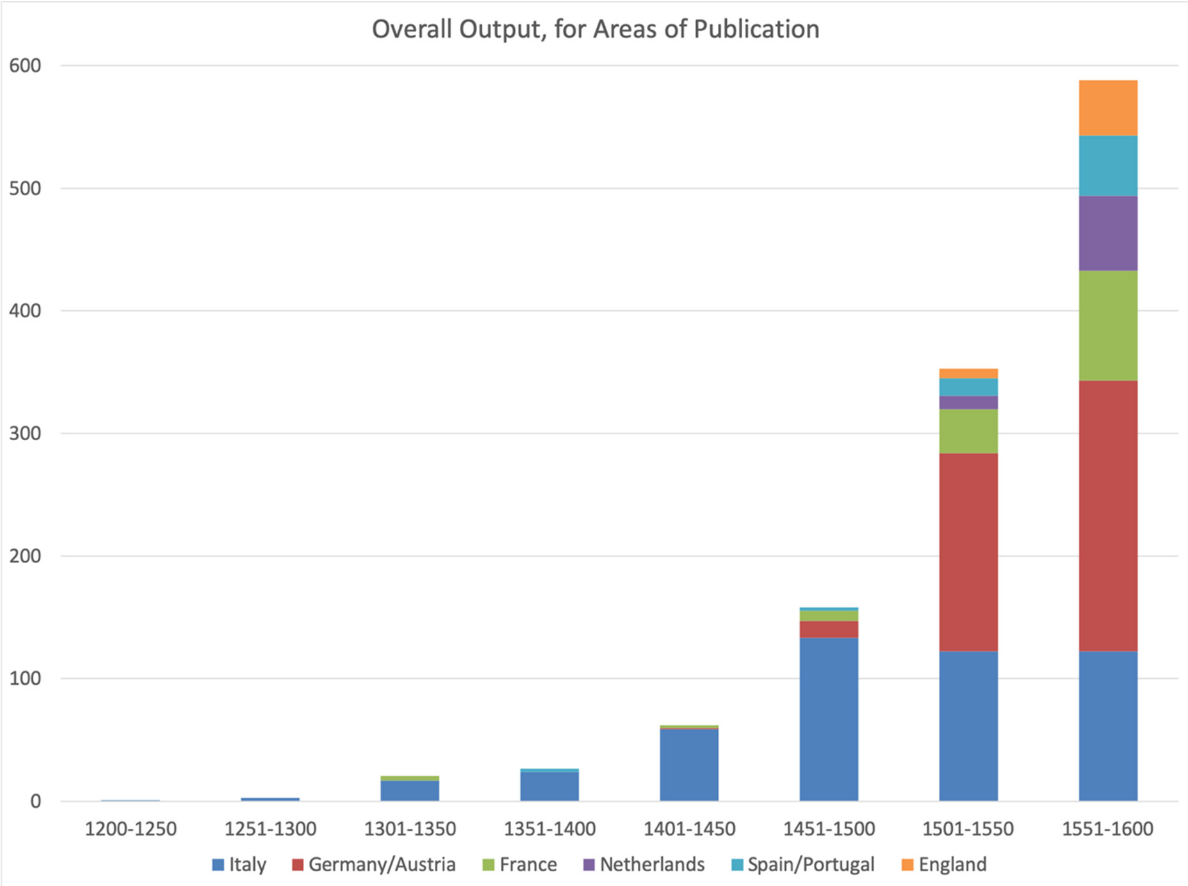


Figure 2 shows the trends in areas of publication using 40 years moving averages. The graph represents absolute values (i.e. not weighted), which makes trends more significant than comparative numbers. Again, the striking feature of the first half of the observation period is the leading Italian production. Interestingly, before the 16th century abacus mathematics shows two periods of growth, namely the period until the middle of the 14th century and the entire 15th century. The second half of the 14th century, instead, is characterized by a slight decline in text production. This could be due to a heaping effect in 1350 (several manuscripts do not report a date, and can therefore only be approximately dated, which increases the probability of text heaping in round dates), or to a decline in text demand.[11] This second option could be due to the demographic collapse in the aftermath of the Black Death.

[11] I am borrowing the concept of the "heaping effect" from (A'Hearn *et al.* 2009).

Figure 2 shows that the 15th century was the golden age of Italian abacus mathematics and suggests that by the end of the century abacus mathematics saturated its diffusion within Italy. This hypothesis is strengthened by the fact that the introduction of the printing press did not determine a marked increase in the production of abacus manuals. After 1478 (the year when the first printed abacus manual is published), the Italian production levels off around an average of 12.5 texts published every five years. Given the higher amount of texts produced in Italy in this period (307 texts between 1478 and 1600), the still high component of manuscript texts (148) and of texts without precise dating (103, both manuscript and printed), the text-heaping effect in this period is stronger. This could explain the two apexes in the beginning as well as in the middle of the 16th century, but does not explain the drop at the very end of the period. In other words, the data suggest a levelling off of the Italian production of practical arithmetic texts in the 16th century, with a decline at the end of the period. An alternative explanation for the decline in the Italian production in this period could again rely on external factors, such as the aftermath of the so-called Italian wars of the 16th century. Overall, the Italian production seems to follow the typical s-shaped curve characterising the diffusion of innovations.[12]

All the other areas show a marked increase in production of practical arithmetic texts, with the printing press as a fundamental driving factor. Germany, the heart of European printing press, stands out as the most important producer of these texts. *Rechenbücher* were a particularly successful genre in 16th century Germany, as shown by the steep increase in the production of these texts. Interestingly, Germany shows a decline in production at the end of the 16th century as well. Very similar trends apply to France, and, with a less marked decline at the end of the period, to Spain. A different trend is instead shown by the latecomers. Among these, the Netherlands and England show a growing production of texts in the end of the 16th century, a period when both areas were laying the foundations for the expansion of Atlantic trade and for their respective golden ages.

[12] As popularized in (Rogers 1962).

Figure 2

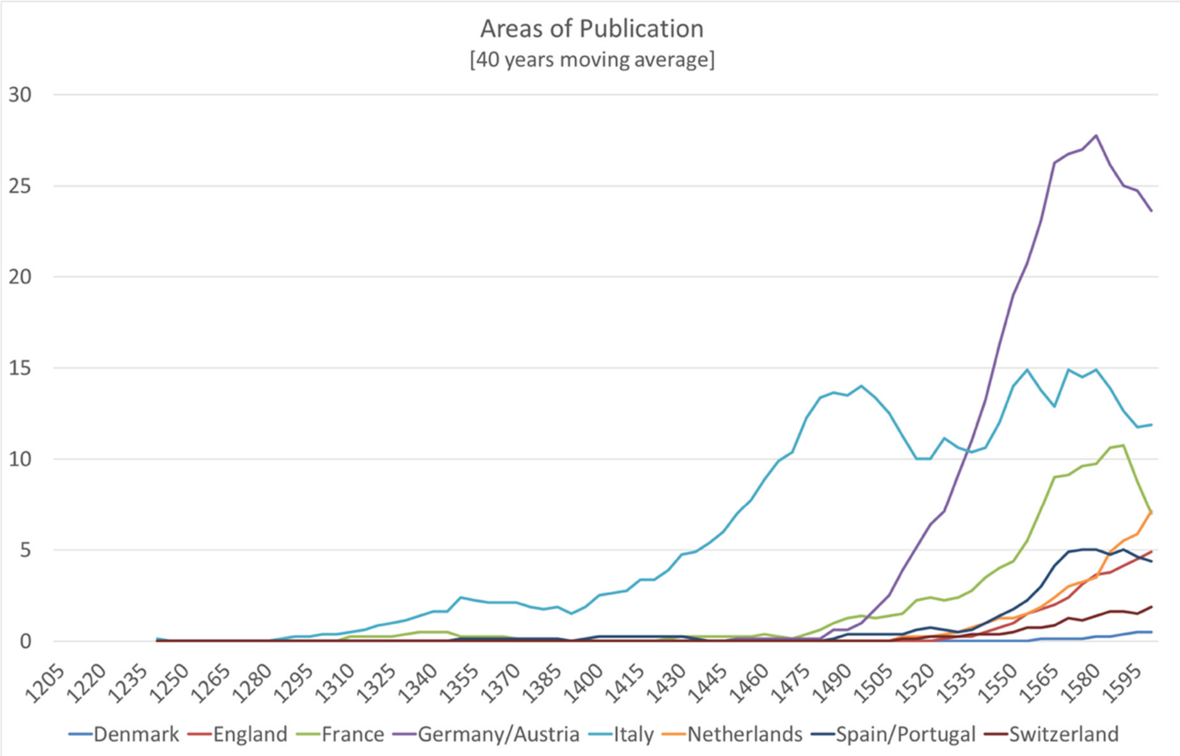
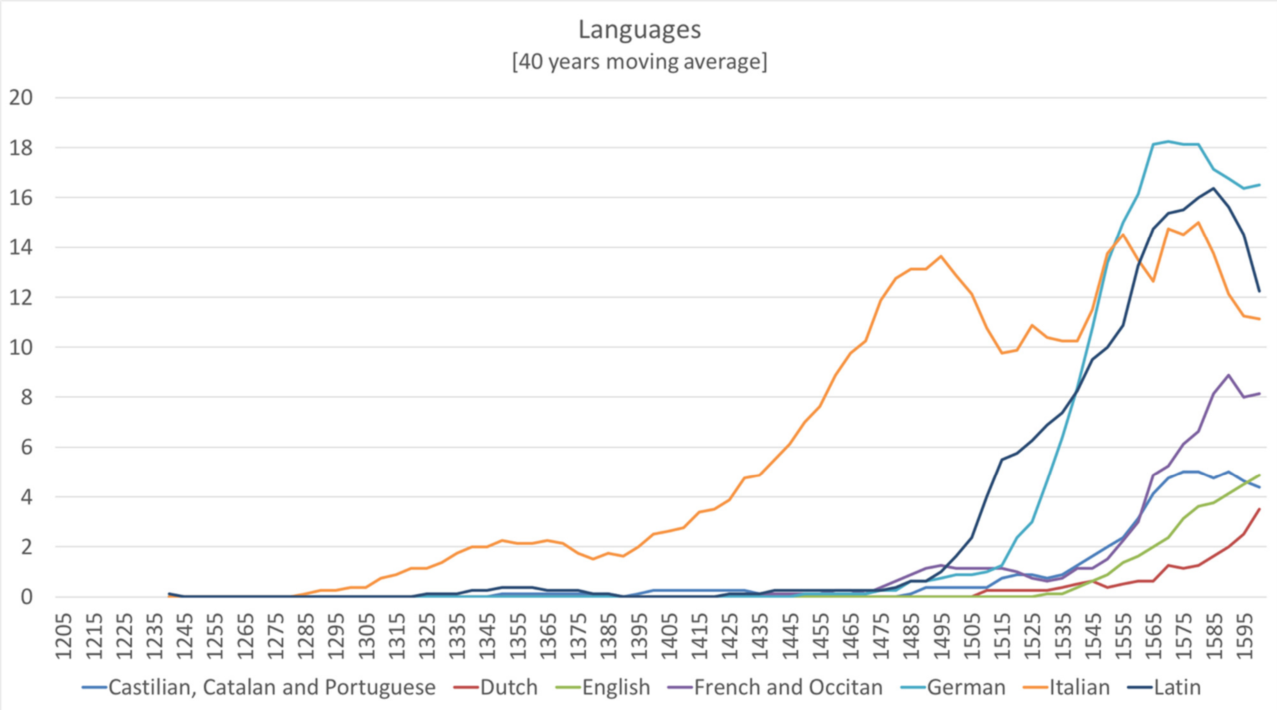


Figure 3 shows language trends, again using 40 years moving averages. The overall picture is similar to the one given in **Figure 2**. What is interesting to notice in this graph is the curve of Latin texts, which cannot be captured by an analysis based on areas. Latin texts tend to be more élite publications, even though in this period, which is also marked by the expansion of humanism north of the Alps, Latin started to circulate also within urban societies. Latin texts are more mobile in terms of geographic span than vernacular ones, as they were published in most of the areas covered by the database. It is interesting to notice that the production of Latin texts was remarkable in size (in absolute terms, it is the second language after German) and that it followed the trend of most continental areas, showing a decline in production at the end of the 16th century. The expanding Dutch and English productions at the end of the period are, in fact, mainly in vernacular.

Figure 3



GIS analysis allows to visualise the spread of practical arithmetic on intertemporal maps.[13] The data have been divided into 50-year intervals until 1450, and in 25-year intervals until 1600. The maps record cumulative data, i.e. they record all the texts published up to the map date. **Figure 4** shows the situation in 1350. We can see the early production of abacus manuscripts in Pisa starting from the late 13th century, together with the earliest text written in Perugia (Franci 2003), (Franci 2015), (Bocchi 2017). Close to Pisa it is possible to spot Lucca, and moving eastwards we find Florence, which will eventually grow to become the capital of the manuscript phase of abacus mathematics. Further north, we can see the first abacus text appearing in Venice. We also find abacus manuals written in Avignon as well as in Montpellier. These texts fully belong to the Italian abacus

[13] 138 recorded texts could not be located with precision, and as a consequence these have been excluded from GIS analysis. Since it is more difficult to have precise information on early manuscript material, relatively more documents without a specific location are concentrated in the first phases of the observation period. More specifically, the dataset records 1 unlocated text in the 13th century (out of a total of 4 texts: 25% of total), 19 unlocated texts in the 14th century (out of 41: c. 46% of total), 72 in the 15th century (out of 225: 32% of total) and 47 in the 16th century (out of 1009: c. 4.6% of total).

tradition, as they were written by Italian (probably all Florentine) merchants who were trading with the Papal court, based in Avignon and supplied by the port of Montpellier.

Figure 5 portrays the spread in 1400. It is possible to notice the growing role of Florence, whose production overtook the Pisan one. In the north of Italy, we see appearing other commercial centres, such as Genoa, Bologna and Milan. With **Figure 6** we move the analysis to the European scale in 1450. It is possible to notice the emergence of two main hubs for the diffusion of abacus texts in Italy (Florence and Venice) as well as the appearance of new centres such as Como, Padua, several other Tuscan cities, and Rome.

Outside of Italy, the earliest examples of non-Italian practical arithmetic texts employing Hindu-Arabic numerals come from the Mediterranean context. The earliest extant manuscripts are from Spain and can be dated between the end of the 14th and the beginning of the 15th century (Rey 2009).[14] In the south of France we find the *Manuscrit de Pamier*, written between 1420 and 1430 in Pamier, a town on the main road between Toulouse and Barcelona (Sesiano 1988).[15] This manuscript preserves an anonymous text written in Provençal which presents the typical contents and structure of an abacus treatise, as it shows how to make operations with integers and fractions, followed by applied problems.

The earliest practical arithmetic text employing Hindu-Arabic numerals north of the Alps is the so-called *Algorismus Ratisbonensis*. This is an interesting text, as it could represent an intersection of the Latin tradition of the *algorismi* and that of abacus mathematics. Written by a Benedictine monk at the monastery of Sankt Emmeran in Regensburg – a Free Imperial city on the trade routes to Italy – in the middle of the 15th century, the *Algorimsus Ratisbonensis* played a similar role in Germany as Fibonacci's *Liber Abaci* had done in Italy (Vogel 1954).

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[14] These are the *Libro de arismética que es dicho alguarismo* (c. 1395), ms. 46 de la Real Colegiata de San Isidoro de León (Potro 2000); the ms. 155, ff. 144r-164r of the Real Academia Española (beginning of 15th c.); the ms. 10106 of the Biblioteca Nacional in Madrid (beginning of 15th c.). The earliest of these texts was probably written in Seville.
[15] Paris, Bibliothèque Nationale, nouv. Acq. Fr. 4140 (NAF 4140).

Venice played a central role for the transmission towards north of practical knowledge in general, and of practical arithmetic in particular. The *Fondaco dei Tedeschi* was a centre where German and Italian commercial communities encountered each other. Starting from the 15th century, we find a growing number of German youngsters moving to Italian commercial centres (and mostly to Venice) to gain an education in commerce or in other practical arts. The international merchant-banker Jakob Fugger, his accountant Matthaüs Schwarz and Albrecht Dürer are just the best-known examples of a wider phenomenon (Braunstein 2016, p. 435).

With **Figure 7** we can observe the situation in 1475. This is still a world of only manuscript sources. We can see more texts appearing in the south of France as well as the first text appearing in Paris. This is Jehan Adam's *L'arithmétique aux jetons* which, despite its title, reports calculations with both the reckoning table and Hindu-Arabic numerals (Thorndike 1926). In Italy, it is possible to notice the rapid expansion of Florence as the leading centre in 15th century abacus mathematics as well as the diffusion of the production of these texts in several other central-northern cities. This is probably the apex of Florence both as financial hub and as capital of abacus mathematics. The most successful abacus treatise of the 15th century – Benedetto da Firenze's *Trattato d'abacho* – is attested for the first time during this period (Ulivi 2002).

With **Figure 8** we move to the end of the 15th century, and we start to see the acceleration in text production triggered by the printing press. In Italy, Florence is still the leading centre, followed by Venice, and the production of abacus treatises is increasingly widespread, with texts published also in Naples, Nola and in the region of Calabria. The earliest printed practical arithmetic was published in this period. This is the so-called *Aritmetica di Treviso*, published in 1478 (Swetz 1987). The second oldest practical arithmetic manual printed in Europe was published in Barcelona. This is the *Suma de la art de arismetica* by Francesc de Santcliment, written in Catalan and printed in 1482, which is based on the model of abacus manuals.[16]

In France, we see an expansion in Paris and the appearing of Lyon, following the royal protection to local fairs granted by Louis XI in 1463. This is the city where Nicolas Chuquet completed in 1484 a manuscript which includes a commercial arithmetic, a collection of problems, a geometry and the *Triparty en la science des nombres*.[17] Chuquet was a sophisticated mathematician, and his *Triparty* has drawn the attention of historians of mathematics because of its original mathematical contributions.[18] However, Chuquet's manuscript also includes the typical contents of abacus mathematics, and is probably based on Italian sources (Flegg *et al.* 1985), (Benoit 1988).[19] Another French source of this early phase is the *Kadran aux marchans* by Jehan Certain, probably completed in 1485 Marseilles.[20] Also in this case, the subjects covered are those of abacus mathematics, and the sources and influences are from the Italian context (Benoît 1981).

Moving towards the German-speaking world, the easternmost point represents Vienna, where the first printed edition of Georg von Peuerbach's *Elementa Arithmetices* was published in 1495.[21] This was a very synthetic manual, covering the main subjects of basic arithmetic in the turn of 8 folios. We can also see appearing several cities in the central region of Germany, among which are Cologne, Strasbourg, Nuremberg, Bamberg, and Leipzig. Of these, Cologne, Strasbourg, Nuremberg and Augsburg were Free Imperial Cities. Leipzig had a consolidated trading history, as its fair, founded in 1190, was one of the oldest in Europe. Starting from the late 15th century, Francone and Swabian commercial centres such as Augsburg and Nuremberg founded public schools which also

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[16] In Catalunia, practical arithmetic employing Hindu-Arabic numerals is often identified using the term 'abba', which clearly echoes the term 'abaco' employed in the Italian tradition. In Santcliment's text, the formulation of the rule of three follows the standard definition of abacus manuals. The text is available online: http://mdc.cbuc.cat/cdm/ref/collection/incunableBC/id/25441

[17] Bibliothèque Nationale, Français 1346. The *Triparty* is the first section of the ms. (fols 1-147), followed by a collection of problems (fols. 148-210), a geometry (fols. 211-262) and a commercial arithmetic (fols. 264-321). The manuscript is available online at http://gallica.bnf.fr/ark:/12148/btv1b9058845h

[18] The most interesting of which is a non-geometrical symbolism for exponents.

[19] This is particularly evident in his commercial arithmetic.

[20] Paris, Bibliothèque de l'Arsenal ms. 2904.

[21] Before returning to Vienna, Peuerbach studied in Italy, where he probably came in contact with abacus mathematics.

offered mathematical teaching. The masters active in these schools (*Rechenmeisters*) produced a remarkable tradition of practical arithmetic texts (*Rechenbücher*) during the 16th century.

Just to quote one example, the most successful of these manuals is Adam Riese's *Rechenung auff der linihen und federn*, first published in 1522 Erfurt. This second *Rechenbuch* by Riese,[22] which included both counter reckoning and Hindu-Arabic calculation, together with a thorough illustration of their commercial applications, reached the impressive number of more than 100 reprints.[23] The spread of these schools, masters and publications in Germany seems to have strong similarities with that of abacus mathematics in Italy two centuries before.

With **Figure 9** we enter the 16th century. In the Iberic peninsula we see for the first time Valencia and Lisbon. The first practical arithmetic text published in Portugal was the *Tratado da pratica darismetyca* by Gaspar Nicolas, printed in 1519. This was quite a successful text, receiving at least 11 editions stretching until the 18th century. In Italy as well as in France we see a consolidation of previous trends. As far as Germany is concerned, we see the consolidation of the first publication centres as well as the introduction of new ones (Nuremberg, Erfurt, Frankfurt an der Oder). Moving further north, vernacular practical arithmetic reached the Netherlands with texts such as the anonymous *Die maniere om te leeren cyffren*, printed by Thomas van der Noot in Bruxelles in 1508. This is one of the first practical arithmetic texts published in the north of Europe, and the first one known written Dutch. A second edition was printed in 1510 in Antwerp by Willem Vorsterman, followed by a French translation titled *La maniere pour apprendre a cyfrer*, published by the same printer in 1529 (Harreld 2007).

Figure 10 takes us to 1550. In Spain, Madrid, Valladolid and Seville appear for the first time. In France, production increases in the two main centres (Paris and Lyon) and we find a publication in Poitier as well. In Italy, it is possible to notice the acceleration of production in Venice, which

[22] Adam Riese first *Rechenbuch* showed only calculations with the reckoning table and was never reprinted.

[23] The popularity of Adam Riese as a Rechenmeister has survived until the present day in the German saying "nach Adam Riese", a common idiomatic expression employed when dealing with basic arithmetic. A modern edition of Adam Riese's *Rechenbuch* was published in (Deschauer 2013).

eventually overtook Florence thanks to its flourishing printing industry. We observe a consolidation of text production in the central part of Germany, with some relevant accelerations. For example, in the span of these 25 years, in Frankfurt am Main, another key hub of German trade, were printed 19 *Rechenbücher*. We also see the first publications in centres in northern German lands. These are Gdansk (1538, 1540), Lübeck (1547) and Hamburg (1549). This lag of northern German cities – which were, nevertheless, relevant centres of commerce – is an interesting feature which will be discussed more in detail below.

Further East, we see for the first time Prague, Wrocław and Krakow, which were all on a trade route with Italy. In the Netherlands, we can see a growing production in Antwerp, where in 1540 appeared the first edition of the *Arithmeticae practicae methodus facilis*. This is a compact compendium of practical mathematics written by the Dutch mathematician, cartographer and physician Rainer Gemma Frisius. This was another very successful work which received more than 60 reprints, with translations in French, Italian and Hungarian.

This is also the period where English production takes off. The oldest completely surviving practical arithmetic text published in England is the anonymous *An Introduction for to lerne to reckon with the Pen and with the Counters*, published in St. Albans in 1536/7 (Williams 2012). Early English texts have continental sources, i.e. both French and Dutch practical arithmetic works (Richeson 1947), (Bockstaele 1960). The turning point for the diffusion of this kind of publications in England arrived in 1543, with the publication of Robert Recorde's *The ground of artes teachyng the worke and practise of arithmetike*, printed in London by Reyner Wolfe.[24] With at least 37 reprints, this is the work that triggered the diffusion of mathematic primers in England.

With **Figure 11** we approach a mature stage of the European diffusion of practical arithmetic. Moving from west to east, we see the first work published in Porto. This is the 1555 *Tratado da arte de arismetica*, the first work published in Portugal to deal with algebra. In Spain we see a growth in

[24] A digitized reproduction of the work is available: http://eebo.chadwyck.com

production in Barcelona, Valencia, Zaragoza, and Seville. We also see the first publications in Burgos, Salamanca, Granada and Alcalá de Henares. In France, it is possible to observe again a growing production in Paris as well as in Lyon. The publication of practical arithmetic texts is by 1575 consolidated in Switzerland, with the first publication in Zurich (a 1565 reprint of Adam Riese's *Rechenbuch uff linien und zyffren*) and 11 texts published in Basel.

Germany presents a consolidation in production in its main centres as well (Cologne, Frankfurt am Main, Nuremberg, Augsburg, Leipzig, Wittemberg), some of which show a marked growth in text production. This is again the case of Frankfurt am Main, where in 50 years were published at least 56 practical arithmetic texts. We can also notice the beginning of a spread in the north of Germany, with Rostock, Szczecin and Königsberg appearing for the first time. In 1560 we also have the first publication of a practical arithmetic text in Denmark. This is the *En ny konstig regne Bog, udi Tal maader oc Vecter, paa lynnerne och met ziffre* by Anders Olsen, which had at least 4 other reprints in Copenhagen. Starting from this period, the already mentioned steep growth in text production in England as well as the Netherlands can be observed. London went from 7 publications in 1550 to 26 in 1575. In the Netherlands we can see the leading role in the diffusion played by Antwerp, where at least 22 practical arithmetic texts were published by 1575, as well as the spread to new cities (Bruges, Gent, Louvain, Amsterdam). In Italy it is possible to notice, together with a well-established diffusion among most northern and central centres, the overtake of Venice on Florence.

Figure 12 brings us to the end of the observation period. The overall picture is that of an incremental growth of already established trends. This applies to most of Europe (Portugal, Spain, France, Germany, Italy) where production stops accelerating and only few new centres appear in this 25-years window. England and the Netherlands stand out of this trend. London is characterised by a marked growth in text production, going from 26 texts in 1575 to 52 texts in 1600. A similar pattern is followed by the Netherlands, where overall production jumps from 32 texts in 1575 to 74 texts in 1600 and where text production spread to several new centres, especially in Dutch-speaking territories. The new centres are Middelburg, Dordrecht, Rotterdam, Delft, Leiden, Haarlem, and Alkmaar. Further east, the diffusion of practical arithmetic reached for the first time Hungarian-speaking lands, with publications in Debrecen and Cluj.

Figure 4

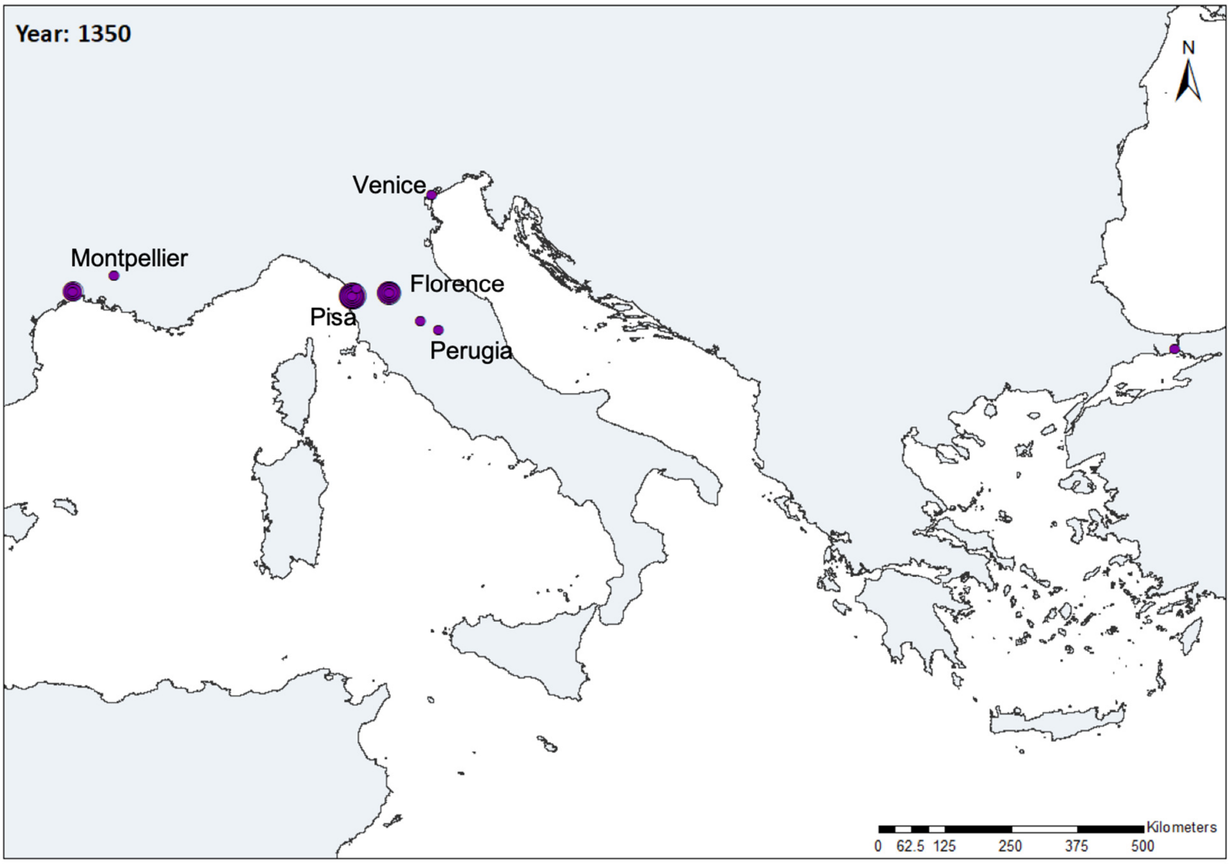


Figure 5

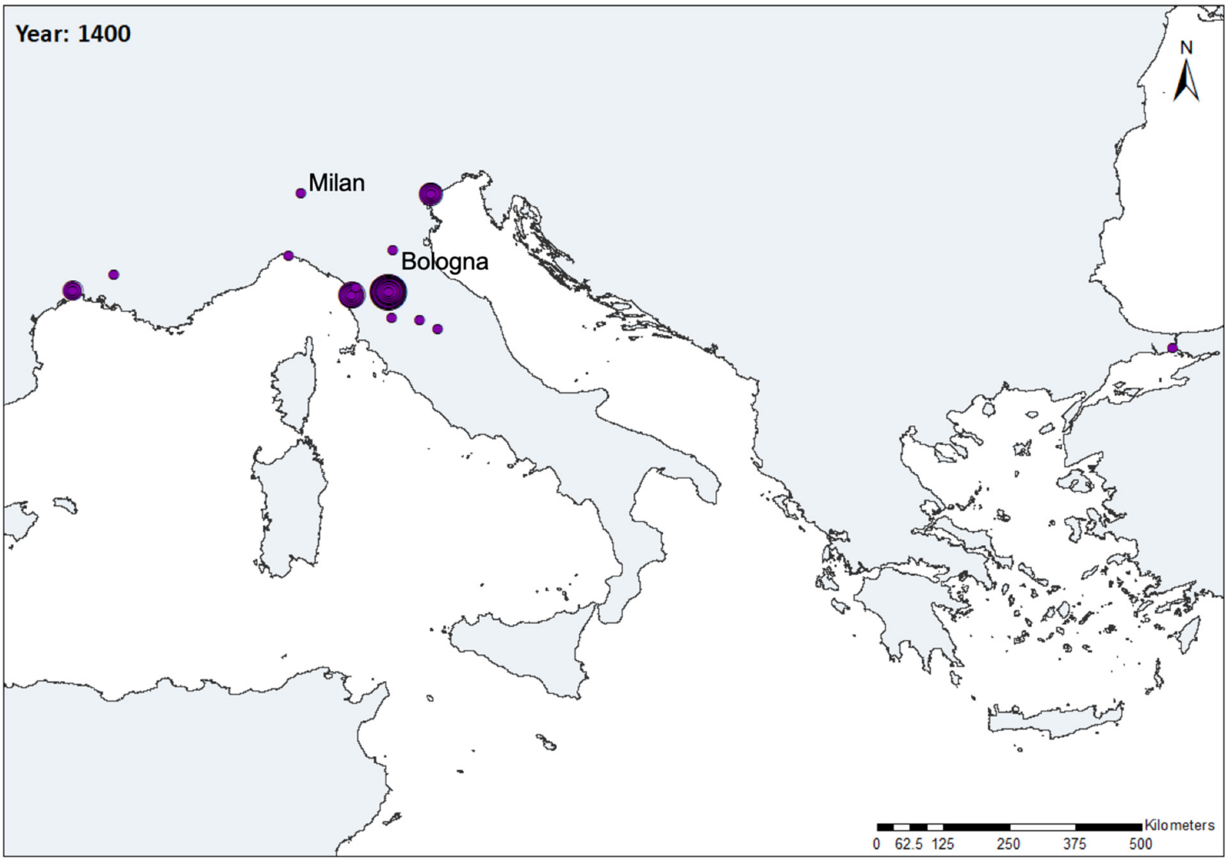
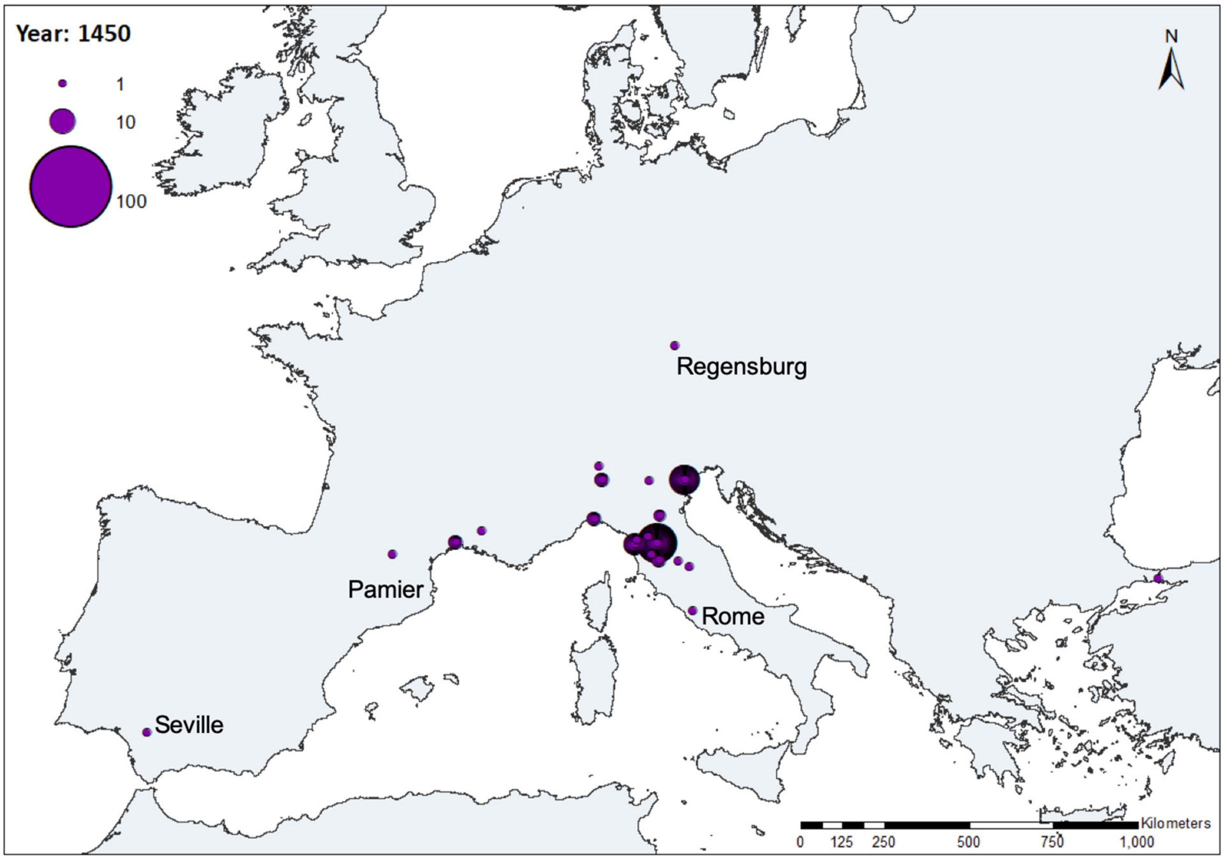


Figure 6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1
10
100
N
Regensburg
Pamier
Rome
Seville
Kilometers
0 125 250 500 750 1,000

Figure 7

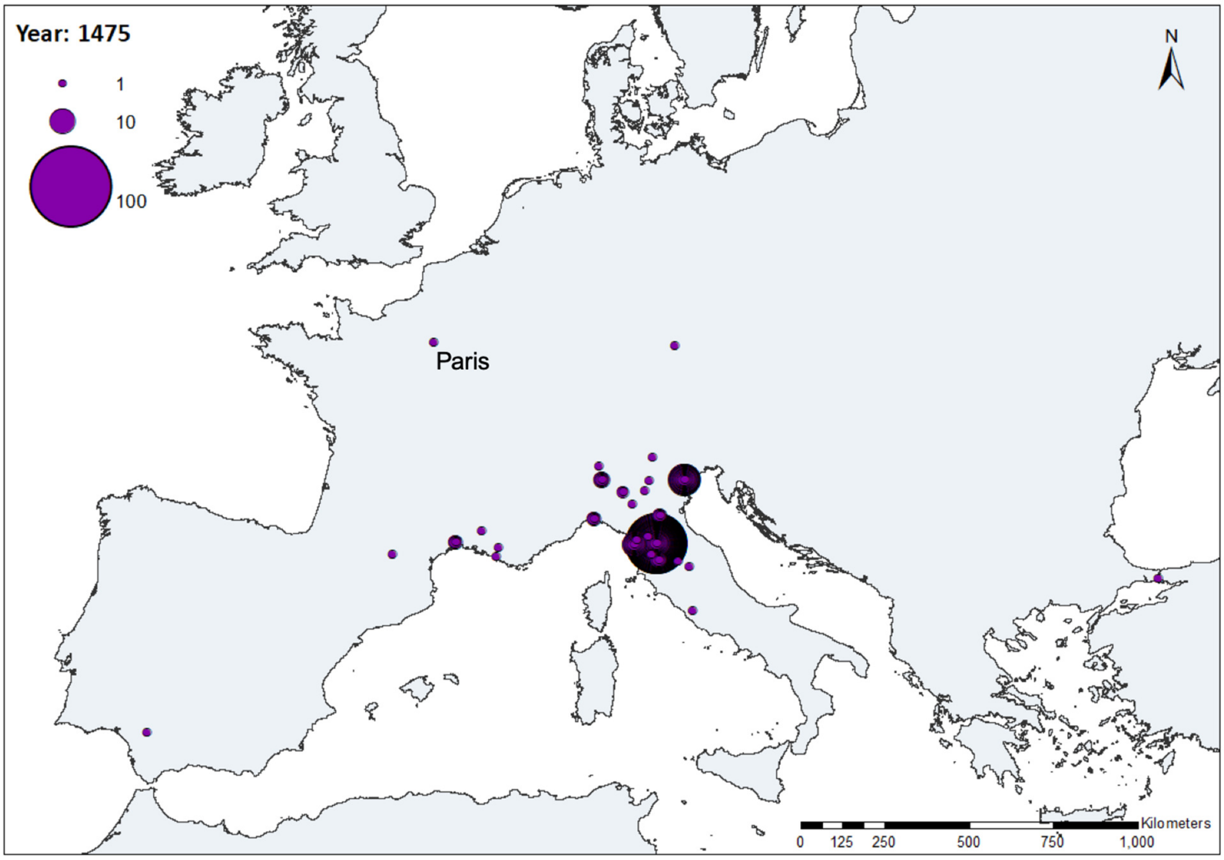

Year: 1475
1
10
100
N
Paris
Kilometers
0 125 250 500 750 1,000

Figure 8

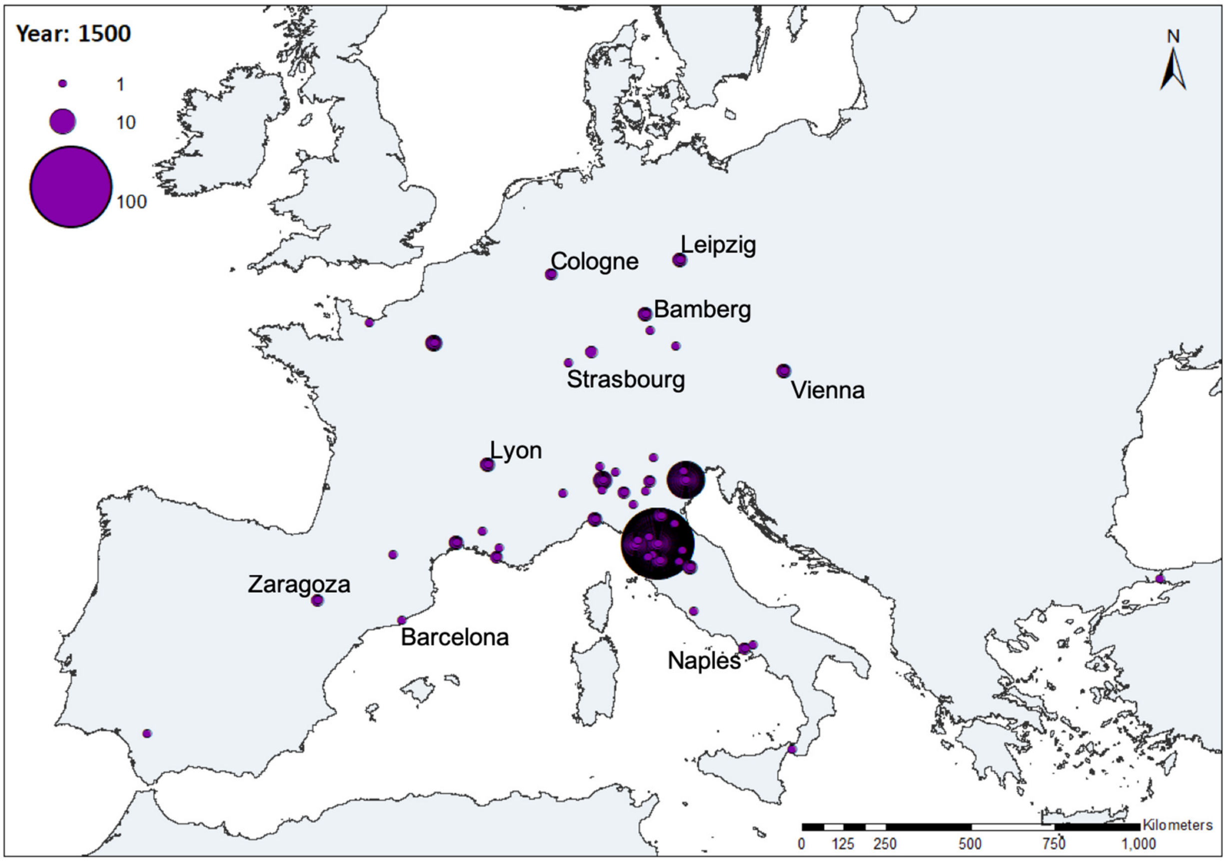


Figure 9

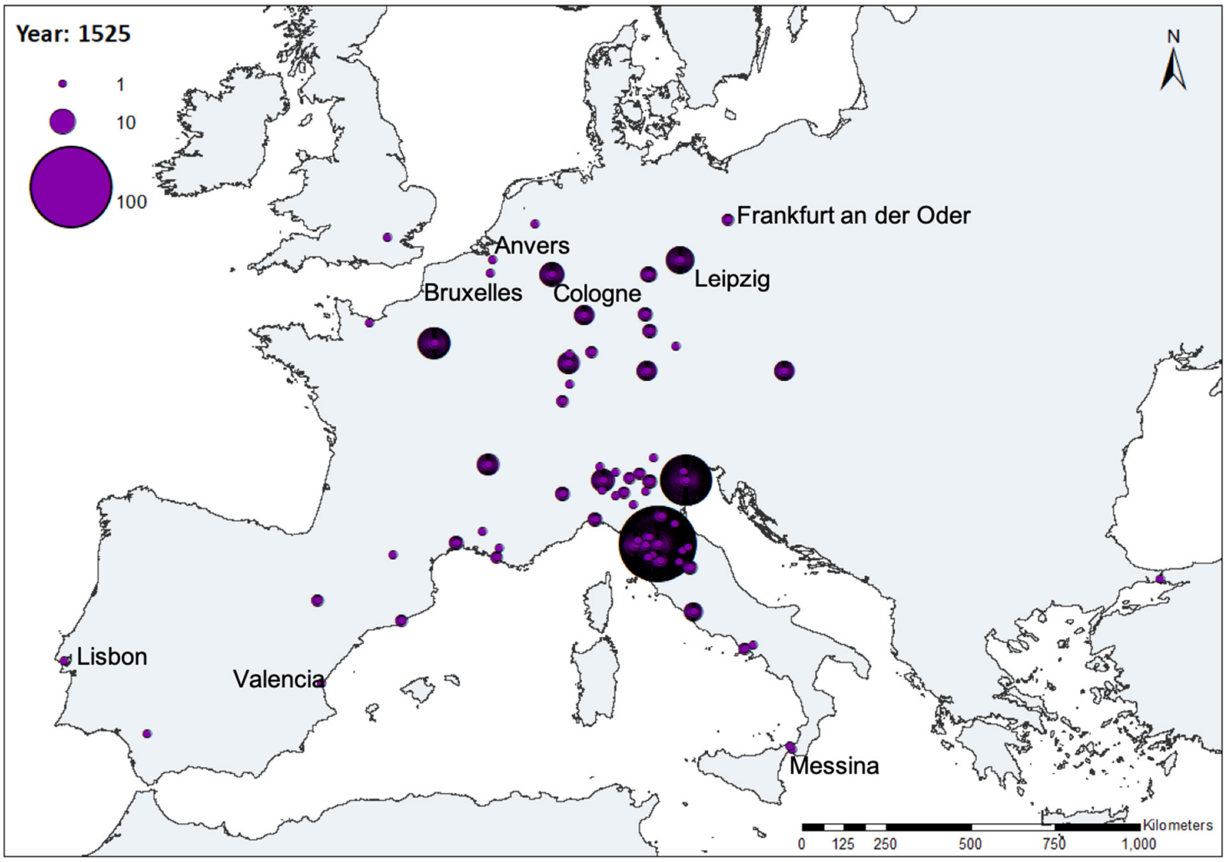


Figure 10

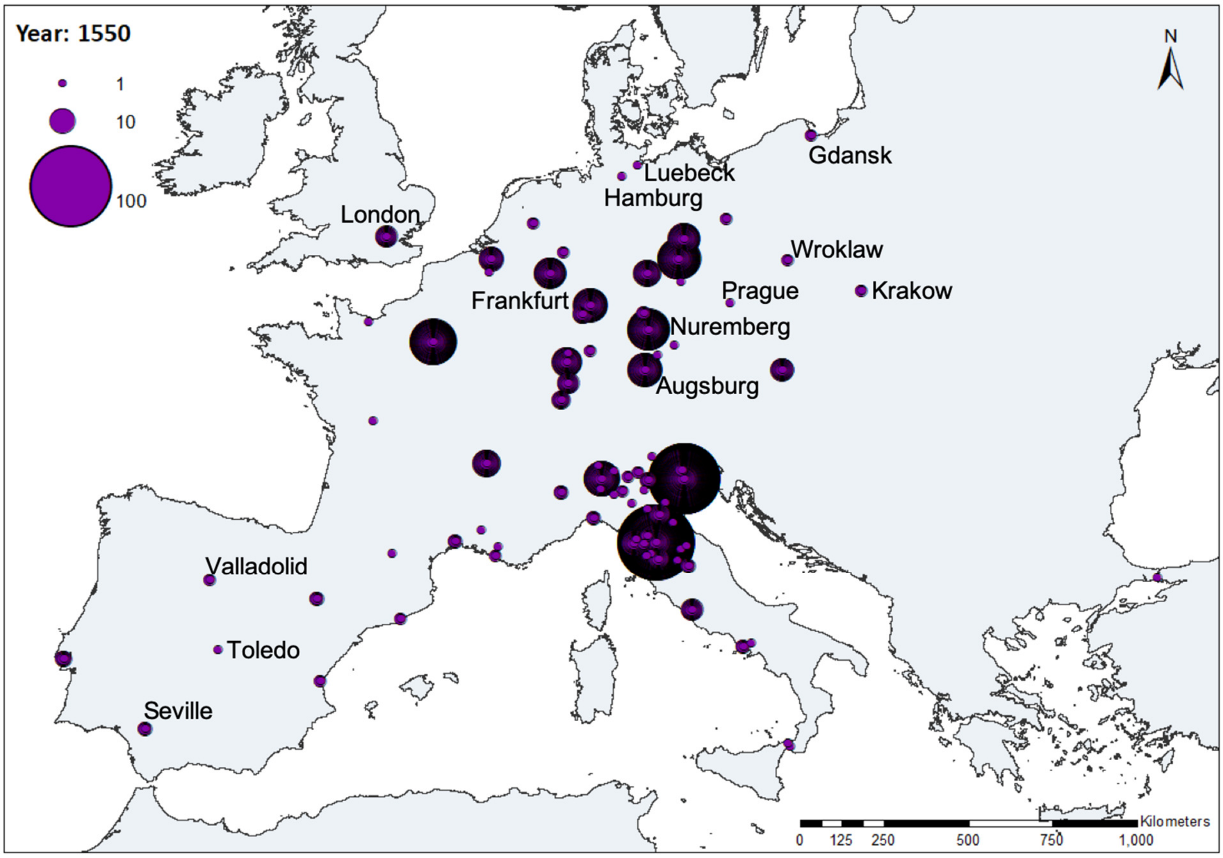


Figure 11

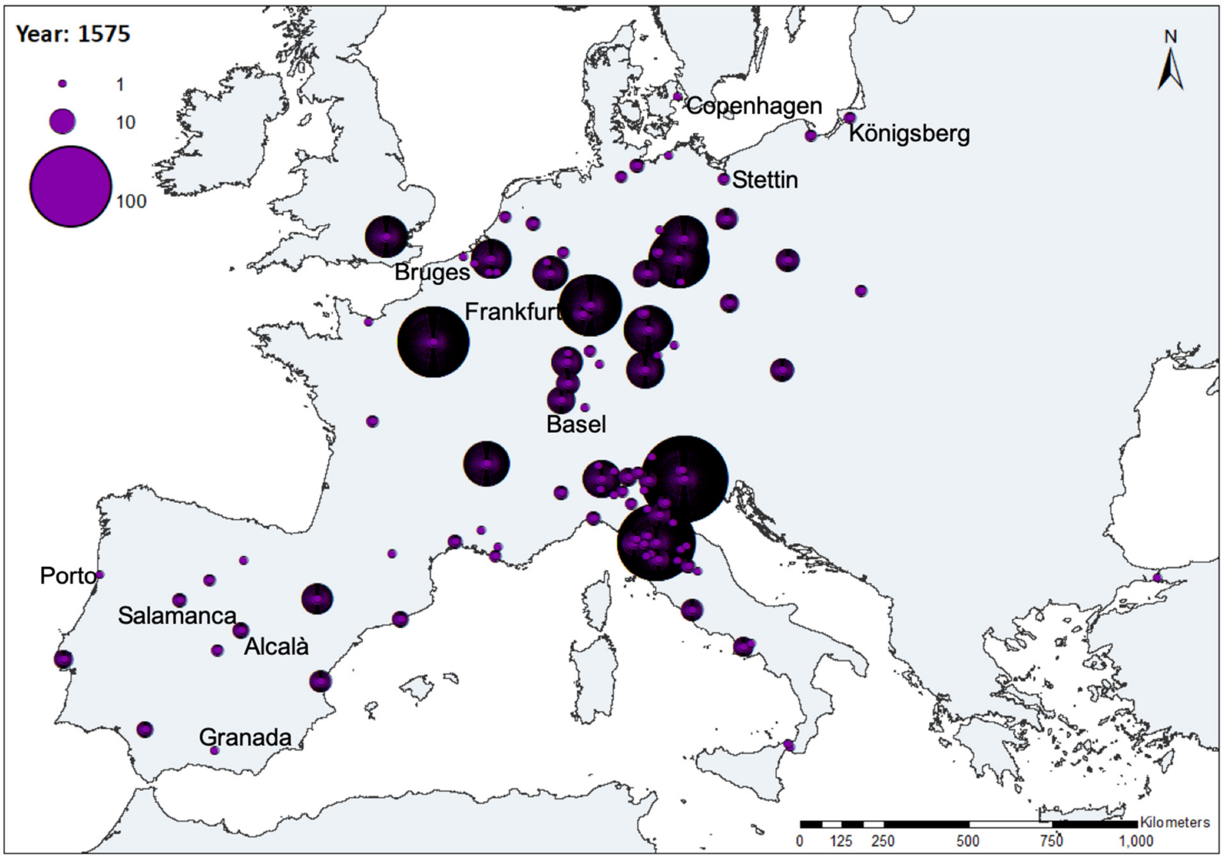
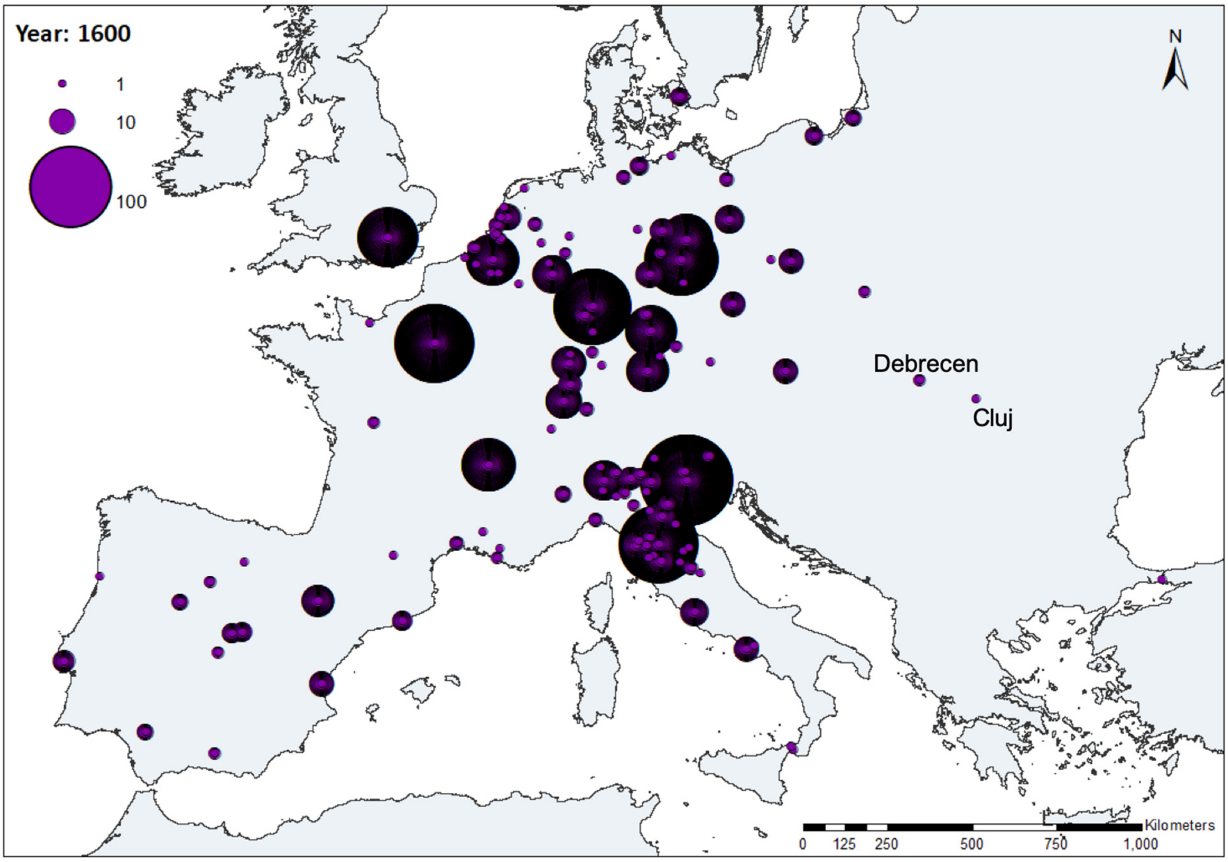


Figure 12



Year: 1600
1
10
100
N
Debrecen
Cluj
Kilometers
0 125 250 500 750 1,000

III.

The previous analysis gives the picture of a continuous tradition addressed to practitioners which spread a set of mathematical tools from the south to the north of Europe. This spread can roughly be divided into three phases. The first phase spans until the late 15th century. This is the period where, with the innovations of the commercial revolution, merchants faced a growing need for advanced mathematical instruments, especially to calculate multiplications and divisions, which led them to adopt Hindu-Arabic numerals. This is also the period where Hindu-Arabic numerals appeared in Italian commercial documentation. The facsimile reproductions of several Italian commercial documents published in (Melis 1972) show that, starting from the 14th century, Hindu-Arabic numerals became increasingly frequent in Italian merchants' accounting practice.[25]

Hindu-Arabic numerals also appeared in Italian public accounts from the 14th century. Among the archives of the *Estimo* of Florence, it is possible to find Hindu-Arabic numerals within the text of single entries from the middle of the century.[26] The ten digits can be found in the registers of the *Prestanze* from the 1380s in the foliation and in draft calculations carried out on spare spaces (such as on the back cover and on the endpapers).[27] Moving to the following century, the most remarkable public accounting enterprise was the famous 1427 *Catasto* (Herlihy & Klapisch-Zuber 1985). The officials who compiled the so-called *campioni* – i.e. the summary reports assessing the wealth of each Florentine household – resorted extensively to Hindu-Arabic numerals. What is more, in the *campioni*

[25] The transcriptions provided in (Melis 1972) do not follow the use of the different numerical notations of the sources, so it is necessary to refer to the facsimile reproductions. Particularly interesting are documents 123-129, 136, 151-152 (reproducing both personal and companies' accounts, where Hindu-Arabic numerals are used to make calculations or to number pages); 132-133, 142 (where Roman and Arabic numerals are used to distinguish between currencies), 155 (where the entries in the *Libro maestro* are recorded in Roman numerals, while the corresponding cheques report values in Hindu-Arabic numerals); 159-160 (where the entries in the *Quaderno dei cambi*, recorded with Hindu-Arabic numerals, are reported in the *Mastro* in Roman numerals).

[26] Florence, Archivio di Stato, Estimo Nr. 6: Registro dei pagamenti degli allibrati del quartiere di S. Spirito (1351-52); Nr. 8: Libro d'entrata dell'estimo Bue Nero, Leon Nero e Ruote (1351-52).

[27] For example Florence, Archivio di Stato, Prestanze Nr. 945: Registro della 19° prestanza di fiorini pel gonfalone scala (1385).

the use of the two numeral systems is reversed in comparison with the previous examples, with Hindu-Arabic numerals having the upper hand.[28]

Between the late 14th and the 15th century we see the first texts published outside of Italy, but never with the continuity or scale of abacus mathematics. The evidence suggests that the use of Hindu-Arabic numerals in commercial practice remained a virtual Italian monopoly until the late 15th century. The role of the printing press in the diffusion of practical arithmetic texts outside of Italy is clear. It may be that a considerable amount of early manuscript material outside of Italy has been lost and that, if it were preserved, one would have observed a smoother acceleration in production outside of Italy. At the same time, though, we have to consider that there was also a linguistic factor at play until well into the 15th century. Since the Italian abacus tradition is overwhelmingly written in the Italian vernaculars and circulated either within private collections or in abacus schools, it was difficult for non-Italian practitioners to have access to it.

The diffusion of practical arithmetic manuals and schools in central and northern Europe followed a generation of merchants and practitioners who moved to Italy to gain an education in the arts or in commerce, who therefore were trained in Italy and were probably familiar with abacus manuals. The booming central European diffusion of practical arithmetic texts could have occurred at the convergence of these two unrelated phenomena, i.e. the presence of central European practitioners trained in Italy and the introduction of the printing press. It may not be by chance that, representing the moment he was appointed as accountant to Jakob Fugger – who in turn had obtained his commercial education in Italy – Matthaüs Schwarz decided to depict himself recording accounting entries in Hindu-Arabic numerals.[29] Previous research also found that the late 15th century is the

[28] Florence, Archivio di Stato, Catasto 1427. Such widespread resorting to Hindu-Arabic numerals is common to the *campioni* of every *quartiere* as well as *gonfalone*.

[29] On Jakob Fugger, (Häberlein 2012). The *Book of Clothes* is edited in (Schwarz & Schwarz 2015). A reproduction of the picture is available: https://upload.wikimedia.org/wikipedia/commons/0/0a/Fuggerkontor.jpg

period where the account books of Augsburg started to adopt Hindu-Arabic numerals (Crosby 1997, p. 115).

The second stage of the spread spans roughly from 1475 to 1525. This is the phase of the first introduction of the printing press, and of the centres of early adoption of practical arithmetic employing Hindu-Arabic numerals outside of Italy. These early adopters are mainly in southern and central Europe: Lisbon, Zaragoza, Valencia, Barcelona, Paris, Lyon, Cologne, Strasbourg, Oppenheim, Augsburg, Leipzig, Vienna. The third phase spans approximately from 1525 to 1600, when we can observe the introduction and consolidation of the spread in northern Europe. By 1550, it is possible to see the early adopters in this region: Anvers, Bruxelles, Deventer, Hamburg, Luebeck, and the initial production in London. In accordance with this trend, in Antwerp Hindu-Arabic numerals started to appear in private and public accounts starting from the middle of the 16th century, and their use became consolidated from the 1580s (Meskens 2013).

The lag between the north and the centre of Europe is particularly evident in Germany and the Netherlands, and is an unexpected finding. Both areas are divided between a southern group of centres adopting Hindu-Arabic methods by 1525, and a northern group of centres adopting them only from the second half of the 16th century. This lag of the northern part of both Germany and the Netherlands, which is apparent also from a linguistic point of view,[30] is surprising, given the fact that this was a developed commercial region. It is possible to speculate as for the reasons why this lag took place. The distinction between Catholic and Protestant cities does not provide neatly correlate with this lag. While it is true that quite a few late adopters (Wittenberg, Szczecin, Hamburg) were

[30] As far as the Netherlands are is concerned, whereas in Anvers (an early adopter) the majority of texts were published in French (20 out of 40 total texts, with 12 Dutch, 6 Latin, 1 German and 1 Portuguese documents), all of the 16 texts published in Amsterdam (a late adopter) were in Dutch. In Germany, all of the texts published in Hamburg, Lübeck, Szczecin, Gdansk and Königsberg (all late adopters) were published in German. In early adopting centres, instead, there is a good mixture of German as well as Latin texts. Of the 22 texts published in Cologne, 20 are in Latin and only 2 in German. In Strasbourg at least 17 works were published, of which 10 were in Latin and 7 in German. In Leipzig it is possible to find 41 German texts and 37 Latin ones, for a total of 78 publications before 1600. Augsburg, however, does not follow this trend: all of its 26 texts were published in German.

cities which turned quickly to the Protestant faith, it is also true that a few early adopters did so as well, such as Leipzig and Strasbourg.

On the institutional level, the most important divide was between Hanseatic and non-Hanseatic cities. Among the adopters in 1500, only Cologne was a Hanseatic city (and a peripherical one), whereas all of the others (Strasbourg, Nuremberg, Bamberg, and Leipzig) had never been members of the League. In 1525 the only Hanseatic city we see is Erfurt (again, a peripherical centre), whereas all of the other centres are not part of the League. The same applies to the Netherlands, where up to 1525 all the recorded cities were not part of the Hansa: Bruxelles, Anvers and Deventer (which had left the League in 1500). Only in 1550 can we see the first publications in Hanseatic centres (Hamburg, Lübeck, Gdansk), which consolidated in the following years, with a substantial lag on other non-Hanseatic cities of the centre of Germany. For the Netherlands, such a lag is even more evident, with the first publication in the Hanseatic territory appearing in 1563 Amsterdam.

This observation is surprising, given the prominent commercial role of Hanseatic cities, and could be a result of the focus of Hanseatic trade on the North and the Baltic sea, rather than on trade with the south. This lag is consistent with research on the history of accounting, according to which the development of accounting techniques was slower in Hanseatic cities (Braunstein 2016, p. 444-448). Further research is needed to tell whether this lag also occurred in with financial instruments. This finding is interesting because it opens the hypotheses of a reluctance among Hanseatic merchants to adopt techniques that were coming from abroad and, arguably, from the competitors. This in turn opens up the question as to what extent commercial communities were aware of their identity and jealous of their economic and commercial competences.

There is anecdotal evidence that goes in this direction. In a similar fashion to double-entry bookkeeping, which was often labelled as 'Italian' in bookkeeping manuals, also practical mathematics employing Hindu-Arabic numerals was perceived as a science coming from the south. It is not rare, among practical arithmetic texts written in German, to define Hindu-Arabic numerals as 'welsch' – a blurred term generally meaning the territories of the ancient Roman world, and widely indicating in this period the Italian language and territories.[31] In any case, this is a complex theme which I do not intend to explore here further, but which is worth mentioning because the recorded evidence could contribute to addressing the question. Moreover, more research is needed to tell whether this gap occurred also in the adoption of new financial instruments, such as the bill of exchange.

England stands out as the latecomer in the publication of practical arithmetics using the positional numeral system. This is reflected in a remarkably late adoption of Hindu-Arabic numerals in mathematical practice. It seems that William Cecil, Secretary of State, Lord High Treasurer, and chief advisor to the Queen, did not know how to reckon with Hindu-Arabic numerals, as in his accounts he consistently transcribed the Hindu-Arabic figures provided by his clerks into Roman numerals (Stone 1949, note 4).[32] Samuel Pepys decided to take private lessons of practical arithmetic in 1662, after his appointment as Clerk of the Acts to the Navy Board (Pepys 1996). The ten figures started to appear in private accounts between the last decade of the 16th and the middle of the following century, while public accounts were more reluctant in adopting the new numeral system (Jenkinson 1926), (Wardley & White 2003), (Froide 2015), (Otis 2017).

IV.

The evidence provided in this paper shows that the transition of European practitioners from Roman to Hindu-Arabic numerals was an innovation cycle which started from the Mediterranean in the late 13th century and took almost 2.5 centuries to complete. The tradition of practical arithmetic allows to open up a perspective on a progressive diffusion of a set of mathematical tools which were first adopted in Italy with the commercial revolution of the 13th century. Primary and secondary sources suggest that the spread of practical arithmetics correlated with the adoption of Hindu-Arabic

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[31] An example is the 1518 *Ayn new kunstlich buech* by Heinrich Ludwig Schreiber (Henricus Grammaticus).

[32] Marvelling at this, Stone concluded that this "industrious economic administrator, perhaps the first statistically minded statesman in English history" was "anachronistic in his mathematical training" (Stone 1949, note 4). The argument may be reversed, and one could contend that Cecil's clerks were rather up-to-date.

numerals in commercial documentation. Even though further research is needed to document in detail when and where Hindu-Arabic numerals were adopted in commercial practice,[33] this correlation suggests that the tradition of practical arithmetic was the driver of the definite adoption in Europe of the positional numeral system. While it is true that the printing press played a key role in the spread of practical arithmetic beyond the Alps, the "radical innovations in business practice" (Dittmar & Seabold 2019) had a considerable circulation in central-northern Italy also in the pre-printing press era.

The fact that the introduction of the printing press did not determine a marked expansion in the Italian production of practical arithmetic manuals, which levelled off during the 16th century, testifies that by the late 15th century the Italian market for these ideas had already been saturated by manuscript production. Moreover, institutional developments in Italian commercial cities anticipated European ones. Abacus schools and abacus masters of the 14th and 15th centuries show strong similarities with German, Dutch and English practical arithmetic schools and masters of the 16th century. Though in a different social and technological context, we have evidence showing that also abacus masters and abacus schools were operating in a competitive market of ideas. Abacus mathematics circulated widely enough in Italian commercial cities at the end of the 16th century that a miller from the countryside of the Venetian state could know it.[34] In this perspective, these texts and institutions were a relevant premise for the process of quantification and rising numeracy that would have also characterised successive developments in the north of Europe (Crosby 1997) (Deringer 2018).

It is important to point out that the spread of practical arithmetic manuals does not represent a story of forerunners or of cutting-edge innovations. Italian merchant-bankers had been active in

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[33] The study of building accounts, which are probably among the best surviving accounting sources, could be an interesting direction to pursue such research.

[34] "Non sappiamo se qui, come in altre località del Friuli, il vecchio sistema della rotazione delle cariche fosse stato sostituito dal sistema elettivo. In questo caso, il fatto di sapere «leggere, scrivere et abaco» aveva dovuto favorire Menocchio" (Ginzburg 1976, p. 4).

northern Europe using Hindu-Arabic numerals, bills of exchange and double-entry bookkeeping for at least two centuries when the first practical arithmetic and bookkeeping manuals appeared there, often flanked by the activity of practical arithmetic masters and schools. The spread of the tradition of practical arithmetic manuals is therefore a conservative estimate for the diffusion of a set of skills associated with the commercial revolution, and should be understood as a proxy to observe their consolidated transmission across commercial communities. Moreover, practical arithmetic was arguably the premise for the diffusion of the other characteristic innovations of the commercial revolution, as a basic understanding of arithmetic was necessary to use tools and techniques such as the bill of exchange and double-entry bookkeeping. In this sense, the spread of practical arithmetic testifies of the diffusion of a wider tacit knowledge, of a set of skills which were not necessarily expressed in the manuals themselves, but which found in practical arithmetic their necessary mathematical premise.

The evidence provided suggests that Hindu-Arabic numerals, and arguably the commercial techniques associated with them, remained a virtual Italian monopoly until the late 15th century. This is in accordance with the fact that the earliest non-Italian international merchant-bankers, such as Jakob Fugger, emerged in this period, as well as with anecdotal evidence. However, further research is needed to document this in detail.

Econometric research has shown that the spread of these ideas in printed media had a causal impact on city growth and a strong correlation with individual achievement in bourgeois occupations in the 16th century (Dittmar & Seabold 2019). For the reasons given above (a remarkable manuscript tradition, dedicated institutions, a competitive market of ideas), such ideas had a relevant impact on economic practice also during the pre-printing press era. This, in turn, suggests that numeracy and human capital played a role in economic development already from the onset of the commercial revolution, as the innovations introduced during that period required a specific training. Following Mokyr's argument that the economy of the Enlightenment "had roots in the commercial capitalism of the later middle ages and the sixteenth century" (Mokyr 2005, p. 339), the analysis of this paper suggests the possibility of extending Mokyr's understanding of 'useful knowledge' to the commercial revolution of the 13th century.

The general picture given by the present analysis is that of a process of transmission of practical knowledge from the south to the north of Europe which followed the opposite direction of successive processes of 'emulation' (Reinert 2011). Furthermore, the analysis shows that practical arithmetic manuals, and the tacit useful knowledge associated with them, did not spread through maritime, but rather through inland routes. If it is true that the adoption of practical arithmetic is a proxy to capture the spread of knowledge and skills that had a significant impact on market development, such spread did not follow the hubs of maritime trade, but rather an inland, incremental and proximity-based network.

This is an unexpected finding, which may raise the question as to whether a relevant contribution to the shift of the core of European trade from the Mediterranean to the Atlantic came not only from international trade, but also from intellectual exchanges, i.e. from the transmission of practical knowledge and skills. In other words, this finding raises the hypothesis that, together with the expansion of market access, the European shift towards the Atlantic was also underpinned by a continuous and tacit transmission the useful knowledge developed with the commercial revolution. Under this point of view, the European tradition of practical arithmetic allows to open up a perspective on an archaeology of the history of European economic thought, as it allows for the identification of a continuous process of transmission of those pre-modern "pieces of purely economic analysis" surfacing in medieval mercantile manuals, which Schumpeter excluded from his *History of Economic Analysis* (Schumpeter & Schumpeter 1955, 151-152). This hypothesis of an intellectual perspective on the foundations of Europe's pre-modern economic development is an open question which goes beyond the scope of the present paper, but it suggests a possible direction for further research.

Data appendix

The database records 1280 texts produced between the *Liber Abaci* and 1600, written by more than 340 authors active in over 130 cities.[35] Of these texts, 342 are manuscripts and 938 are printed documents. The recorded languages are Castilian, Catalan, Czech, Danish, Dutch, English, French, German (when possible distinguishing for Low German), Greek, Hebrew, Hungarian, Italian (with several regional variations), Latin, Polish, Portuguese, and Provençal. The areas covered by the dataset are Austria, Bohemia, Denmark, England, France, Germany, Greece, Hungary, Italy, the Netherlands, Poland, Portugal, Russia, Spain and Switzerland.

The database is grounded both on secondary and on primary sources. Among the secondary sources, the most important are obviously catalogues. As far as the Italian abacus tradition is concerned, I have included all the evidence recorded in the most comprehensive catalogue on this tradition (Van Egmond 1980). This catalogue – which is particularly thorough on manuscript sources – does not provide summaries of contents for printed books. Every time it was possible, I have therefore consulted such sources (either direct inspection or through digitised copies) and I have recorded their contents. I have also integrated the evidence provided by Van Egmond with the several findings of successive research in the field.[36]

As far as the European tradition is concerned, I relied on a variety of sources. (Smith 1908) is the foundation of the data on the European scale. This monumental work, which is based on a remarkable effort of archival research, had to be addressed with precise criteria as it includes texts which cannot be considered part of a European tradition of practical arithmetic. The first criterium has been excluding works not presenting Hindu-Arabic numerals. Secondly, among the works employing Hindu-Arabic numerals, only the ones which could be considered as belonging to the tradition of practical arithmetic have been included. This means that the early modern reprints of classical as well as early medieval sources have not been included.

[35] Several texts are written by anonymous authors, hence the estimate on their number.
[36] Such as (Bocchi 2017), (Franci 2015), (Ulivi 2015), (Ulivi 2011), (Franci 2003), (Ulivi 2002).

This criterium ruled out all the reprints of the *Arithmetica* by Boethius and all the early modern works belonging to such a tradition, together with texts dealing with numerology and the *computi* for the calculation of the calendar.[37] Thirdly, only works which could have been relevant to practitioners have been included. This means that only texts showing practical applications of mathematics, or that are explicitly addressed to practitioners, or that could have been used as part of a commercial training, have been included. This, in turn, means that purely theoretical works developed within universities, as most of the tradition of the *algorismi*, have not been included. Algebraic and geometric texts have been included only if they presented practical applications (i.e. purely theoretical texts in these fields have been excluded). It is necessary to point out that, however, since the primary material is not clearly divisible according to strict categories, these criteria were assessed against the singular characteristics of the individual texts, which led to the inclusion of some theoretical works and the exclusion of some practical ones.[38]

The information provided by Smith has been checked, depending on the kind of text, on the relevant online repositories. The main ones I have consulted are the Incunabula Short Title Catalogue (ISTC), the Universal Short Title Catalogue (USTC), the English Short Title Catalogue (ESTC), the online catalogue of the Bibliothèque Nationale de France and their digitalisation project (Gallica), the Münchener DigitalisierungsZentrum of the Bayerische Staatsbibliothek, and the Biblioteca Virtual Miguel de Cervantes (Cervantes Virtual). Such research tools, which were not available neither to Smith nor to Van Egmond, allowed to expand considerably the available information on every text, to identify new texts by the same author as well as to identify new authors, to record editions that are missing in previous catalogues, holding institutions and classmarks of original

[37] The *computi* were texts showing how to calculate the calendar and were widely employed, especially within the Church, for the calculation of specific dates, such as Easter. Hindu-Arabic numerals spread also through this tradition, which in some cases considerably preceded the spread of practical arithmetic. See (Nothaft 2014).

[38] Just to give a few examples, the already mentioned *De arte supputandi libri quattuor* by Cuthbert Tonstall (London, 1522) has been included despite its primarily theoretical focus, given its importance as the first work entirely dedicated to arithmetic published in England. In a similar way, Petrus Ramus' *Arithmeticae libri tres* (first ed. Paris, 1555) has been included because of the importance given by Ramus to applied mathematics (Angelini 2008). Examples of texts employing Hindu-Arabic numerals which, despite being of evident practical relevance, have not been included are the manuals dealing with finger reckoning.

copies, and to report every digital reproduction identifiable. When referring to these tools, I have tried to avoid circularity as much as possible, i.e. to include their evidence only if based on information richer or independent from the sources I was already consulting.

This approach (i.e. the stated criteria followed by the consultation of online repositories) has been adopted when dealing with all consulted catalogues. The works quoted in Smith (1939) have been included only if they had clearly been consulted by Smith himself or if the USTC provided more information on them. Navarro Brotons (2000) provided precious information on Iberic sources, while (Hoock *et al.* 1991) allowed to consolidate the evidence for central and northern Europe. The information provided by these main catalogues has been integrated resorting to a variety of papers and local studies.

Furthermore, I have paid visits to archives in Florence and in Bologna, as well as to archives in the British Library, the University Library of Cambridge and the Bibliothèque nationale de France in Paris, in order to gather first-hand evidence and documentation on early and particularly interesting primary sources of Italian and European practical arithmetic traditions. Every time it was possible, I have checked the contents of the texts. At the present stage of development, I have recorded the contents of 1051 texts (82% of total), collected either through direct inspection of original or digitised copies, or relying on secondary studies.

As far as the representativeness of the sample is concerned, I can only give tentative considerations. The main issue is of course that the size of the population (of which the database is a sample) is unknown. In other words, it is difficult to estimate the representativeness of the database because we do not know how many practical arithmetic manuals were actually written in Europe between the 13^{th} and the end of the 16^{th} century. What is possible to say is that, for the early stages of the spread, the Italian abacus tradition surely is the best documented case. The same does not hold for Spain and the south of France, where the known manuscript material is in far smaller quantity than the Italian case. As we have seen, it is possible that several sources from the early stages of the (mainly hypothetical, due to the lack of evidence) Mediterranean-wide mathematical exchanges did not survive to the present day. In other words, the manuscript material of the first stages of the tradition of European practical mathematics may be lost, the best documented case being the Italian one. Among the known surviving manuscript sources of the pre-press age, the database offers a good coverage.

Entering the age of the printing press, the quantity and quality of available sources sensibly increases. Because of the overwhelming amount of the material produced, I have surely missed some texts, but I have resorted to the most reliable repositories currently available and I have tried, every time it was possible, to double-check the recorded information.[39] The area on which the database is arguably weakest are printing-press-era manuscript texts from outside of Italy, since these sources have not been thoroughly studied yet. Nevertheless, as far as European printed texts are concerned, known sources can probably be considered as a representative sample of their respective traditions, and the database provides a good coverage of them. As far as Italy is concerned, the data can arguably be considered representative of both the manuscript and printed traditions.

In other words, as far as the Italian tradition is concerned, both manuscript and printed traditions have been studied and are quite well preserved. The database offers a good coverage of both. Moving to the European level, limited early manuscript material has survived, but it has been often studied in detailed and has been recorded. Printing-press-era manuscript material, on the contrary, has not been thoroughly studied and, as a consequence, is poorly recorded in the database. On the other hand, printed European material is abundant, can be considered as a representative sample of its tradition and has been recorded in the database as thoroughly as possible. For all of these reasons, I believe the database can be considered as a representative sample of the known sources. In my analysis I focus on the periods and areas where the evidence is stronger.

[39] Even if the resources used to compile the database cover most of the European area, I believe that the collected data is most solid for German, Italian and English printed texts. This is due both to the quantity of studies published on these areas (Italy) and to the quality of the work of cataloguing and digitisation of primary sources (Germany and England).

Following (Van Egmond 1980), every manuscript document has been considered as an independent text. Following (Hoock *et al.* 1991), printed texts have been recorded at the book-edition level, which means that every reprint has been considered as an independent text. Among printed documents, the first editions, without counting translations, are at least 310.[40] For every text, I have recorded the following details:

- Year of publication (dated vs estimated)
- Area (ex: Italy, France, England)
- City
- Author
- Title
- Language
- Manuscript vs printed document
- (number of ms. copies)
- (number of reprints)
- (printer)
- Source of the recorded information (both secondary and primary)
- (link to digitized document)
- Holding institution, (classmark)
- Table of contents
- Notes

The main focus of the tradition of practical arithmetic is not on mathematical theory, but rather on its practical applications. As a consequence, practical arithmetic manuals do not follow the deductive structure of the Euclidean model, and are mostly organised into thematic sections comprising lists of solved problems. Theoretical explanations, when present, are given a limited space. Mathematical rules, when present, are generally given without a demonstration at the beginning of their section, and are followed by a list of solved problems where the rule is applied. All the recorded texts employ Hindu-Arabic numerals and rely extensively on the "rule of three", i.e. the calculation of the unknown in a proportion where three terms are known. The rule of three is used in

[40] Sometimes it is not possible to tell whether a document is a first edition. 310 is the number of the documents clearly printed for the first time.

a wide variety of commercial problems, such as exchange, conversion, interest rates, divisions of shares, alloying, barter, etc.

Another defining characteristic of practical arithmetic texts is that of including almost exclusively denominate numbers, i.e. numbers used to count things (units, weights, measures, etc.). Practical arithmetic texts are interested in pure numbers (i.e. ratios) as long as these can be used to calculate a denominate number. This is how the rule of three is universally used in these texts: the focus is not on discovering ratios between entities, but rather, given the ratios, on calculating the values of those entities. This is the main feature that separates this mathematical tradition from the academic one, mainly concerned with pure numbers.

From a theoretical point of view, the main contributions of the tradition of practical arithmetic were the development of the algorithms we still learn to carry out the fundamental operations with Hindu-Arabic numerals and the first developments of modern algebraic notation. For example, Johannes Widmann, who was the first to give a lecture about algebra in Germany, was probably also the first to use in print the signs + and – as symbols of excess or deficiency, applying them to warehouse measures.[41] Robert Recorde was instead the first to use the symbol = as the equals sign.[42] Apart from these developments, its theoretical contribution was rather limited.

As shown in the paper, however, the fundamental role of practical arithmetic was to drive the adoption of Hindu-Arabic numerals in Europe. From this point of view, European practical arithmetic can be thought of as a slow process of vulgarisation of the mathematics introduced in the times of Fibonacci. As such, it spread a package of mathematical theory and mathematical applications which can be summarised in the following subjects (Franci 2003):

- Introduction of the positional numeral system
- Operations with integers
- Operations with fractions
- Rule of three (proportion with one unknown term)
- Rule of false position (a method to solve equations with one unknown)
- Mercantile problems (monetary systems, money exchange, weights and measures, division of profit/losses, barter, interest rates, alloying)

[41] Johannes Widmann (1489) *Behende und hübsche Rechenung auff allen kauffmanschaft*, Leipzig: Konrad Kachelofen.
[42] Robert Recorde (1557) *The whetstone of witte*, London: John Kingston.

- Practical geometry
- Recreational problems
- Algebra

Even if these are the main topics covered by this tradition, not all texts present all of them. There is a high degree of variation among the recorded texts: from manuscript collections of solved problems preserved in personal notebooks to systematic printed treatises. **Table 1** summarises the distribution of contents in the recorded texts, distinguishing for typology (manuscript vs print) and for periodisation (pre- and post-1500). These statistics show the strong effect of the technology of the printing press in the specialisation and standardisation of the tradition. Since the manuscript text is a more flexible medium, manuscript material is more heterogeneous and less systematic. The vast majority of manuscript texts deal with the general subject of 'commercial problems'. In comparison with printed texts, manuscript sources are more likely to show a variety of subjects close but not strictly pertinent to practical arithmetic, such as weights and measures, algebra, geometry, recreational problems and astronomy.

Printed manuals, on the contrary, consistently show higher percentages of more specific contents, such as sections solely dedicated to illustrating the positional numeral system, how to carry out the fundamental operations, or to a separate treatment of the rule of three. This growing standardisation is at the origin of modern mathematical primers. Subjects such as weights and measures, algebra and geometry evolved into specific printed traditions independent of practical mathematics.

***Table 1* – Distribution of manuals' contents, divided for typology (manuscript vs print) and for periodisation**

	All documents		Manuscript		Print		Pre-1500		Post-1500	
Total observations	1051	100%	283	100%	768	100%	222	100%	829	100%
Numeration	644	61.27%	27	9.54%	617	80.34%	34	15.32%	610	73.58%
Operations	792	75.36%	112	39.58%	680	88.54%	94	42.34%	698	84.20%
Rule of three	694	66.03%	69	24.38%	625	81.38%	63	28.38%	631	76.12%
Fractions	709	67.46%	145	51.24%	564	73.44%	126	56.76%	583	70.33%
Commercial problems	665	63.27%	244	86.22%	420	54.69%	188	84.68%	477	57.54%
Alloying	293	27.88%	29	10.25%	264	34.38%	37	16.67%	256	30.88%
Algebra	220	20.93%	90	31.80%	130	16.93%	77	34.68%	143	17.25%
Geometry	195	18.55%	127	44.88%	68	8.85%	111	50.00%	84	10.13%
Recreational problems	167	15.89%	79	27.92%	88	11.46%	67	30.18%	100	12.06%
Measures/weights	167	15.89%	64	22.61%	103	13.41%	45	20.27%	122	14.72%
Calendar/Astronomy	69	6.57%	47	16.61%	23	2.99%	43	19.37%	26	3.14%

'Numeration', 'operations, 'rule of three' and 'fractions' stand for the presence a section specifically dedicated to the illustration of such topics.

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